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BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VIII Numéro 9

VARSOVIE 1960

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie.

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BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

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VOLUME VIII
NUMÉRO 9

VARSOVIE 1960

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, W-WA. MIODOWA 10

Nakład 1056+164 egz.	*Rękopis dostarczono 25.VIII.1960*
Ark. wyd. 7,25, ark. druk. 6,25+0,375	*Podpisano do druku 23.XII.1960*
Papier druk. sat. 80 g kl. III	*Druk ukończono w grudniu 1960*
Format B5, 70×100 cm	*Zam. prod. 340/60 Cena zł 20.—*

WARSZAWSKA DRUKARNIA NAUKOWA, W-WA, ŚNIADECKICH 8

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

Operations and Linear Functionals in Spaces of φ-integrable Functions

by

W. ORLICZ

Presented on June 13, 1960

1. Throughout this paper we shall use without recourse the notation of [2]. Let (a, b) be finite and $M(a, b)$ the space of measurable and almost everywhere bounded over (a, b) functions with usual norm $\sup^*_{(a,b)} |x(t)|$ *). In the case of $a = -\infty$, $b = \infty$, $M_0(a, b)$ will be the space of functions almost everywhere bounded, vanishing outside a certain finite interval. *γ-convergency* in $M_0(a, b)$ will be defined then as follows: $x_n \in M_0(a, b)$ is *γ-convergent* to x_0 if x_n vanish outside a certain finite (c, d) and $x_n(t) \to x_0(t)$ uniformly in (c, d). We similarly define $M_0(a, b)$ and *γ-convergency* in the remaining cases of infinite intervals. We denote by U a distributive operation over $L^{*\varphi}(a, b)$ assuming values from an F or B-normed space; let further $\xi, \eta, \ldots$ denote distributive functionals over $L^{*\varphi}(a, b)$. U is said to be *φ-continuous* at x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $(+)$ $I_\varphi(x - x_0) < \delta$ implies $(++)$ $\|U(x) - U(x_0)\| < \varepsilon$. A *$\varphi$-continuous* operation U is called φ-linear. If for every $U \in \mathcal{U}$ $(+)$ with δ independent of U implies $(++)$ then the operations of the family $\mathcal{U}$ are *equi-φ-continuous* at x_0. Let (a, b) be finite. U will be called *absolutely continuous* (in $L^{*\varphi}(a, b)$) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $(*)$ $|E| < \delta$, $E \subset (a, b)$ implies $(**)$ $\|U(x_E)\| < \varepsilon$ ($x_E = x\chi_E$, where χ_E is the characteristic function of the set E). If for every U from a certain family, $(*)$ with δ independent of U implies $(**)$ (x) then the operations of this family are said to be *equi-absolutely continuous*. In the case of an infinite (a, b) U is called absolutely continuous if it possesses the above property plus the following one: for every $\varepsilon > 0$ there exists a positive integer m such that taken any measurable E we have $\|U(x_{(-\infty, c) \cap E})\| < \varepsilon$ for $c \leqslant -m$ (if $a = -\infty$), $\|U(x_{(d, \infty) \cap E})\| < \varepsilon$ for $d \geqslant m$ (if $b = \infty$). Also *equi-absolute* continuity of the operations of a certain family can be similarly extended to the latter case.

1.1. Let U_n be absolutely continuous operations with values from an F-normed space and let

$(*)$ $\qquad U_n(x) \to U(x)$ for $x \in L^{*}\varphi(a, b)$.

) sup = essential lower upper bound.

On the above assumptions: (α) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|E| < \delta$, $E \subset (a, b)$, $I_\varphi(x) < \delta$ imply $\|U_n(x_E)\| < \varepsilon$ for $n = 1, 2, \ldots$ (β) if (a, b) is infinite then, in addition to it, for every $\varepsilon > 0$ there exist positive integer m and $\delta > 0$ such that $\|U_n(x_{(-\infty, c)\cap E})\| < \varepsilon$ for $c \leqslant -m$ (if $a = -\infty$), $\|U_n(x_{(d, \infty)\cap E})\| < \varepsilon$ (if $b = \infty$) for $d \geqslant m$, with $n = 1, 2, \ldots$ if only $I_\varphi(x) < \delta$.

Suppose that there exist certain $\varepsilon_0 > 0$, sets E_n and elements x_n such that $|E_n| \to 0$, $I_\varphi(x_n) \to 0$, $\|U_n(y_n)\| > \varepsilon_0$ where $y_n = (x_n)_{E_n}$. We define by induction the increasing sequence of indices k_n so that the following conditions hold for $n = 1, 2, \ldots$ $I_\varphi(x_{k_n}) < 1/2^n$, $\|U_{k_n}(\bar{y}_{k_n})\| \geqslant \varepsilon_0/2$, where $\bar{y}_{k_n} = (x_{k_n})_{A_n}$, $A_n = E_{k_n} - \bigcup_{\nu=n+1}^{\infty} E_{k_\nu}$. Let $\eta = \{\eta_n\}$ be an arbitrary sequence taking values 0,1. Sets A_n being disjoint we define the function x_η as $\eta_n \bar{y}_{k_n}(t)$ for $t \in A_n$ and 0 for other t from (a, b). It is obvious that $I_\varphi(x_\eta) = \sum_{\nu=1}^{\infty} \eta_\nu I_\varphi(\bar{y}_{k_\nu}) \leqslant \sum_{\nu=1}^{\infty} 1/2^\nu = 1$ and from absolute continuity of U_n it follows that $U_n(x_\eta) = \sum_{\nu=1}^{\infty} \eta_\nu U_n(\bar{y}_{k_\nu})$.

Since $U_n(x_\eta) \to U(x_\eta)$ for every sequence taking 0,1 it follows that $\|U_{k_n}(\bar{y}_{k_n})\| \to 0$ (comp. [1]) and we come to a contradiction.

Proof of (β) runs similarly.

Having 1.1 we show:

1.2. If U_n are absolutely continuous operations taking values from an F-normed space and if for every $x \in L^{*\varphi}(a, b)$ $U_n(x)$ is bounded then: (α) for every $\varepsilon > 0$ there exists $\delta > 0$ such that from $|E| < \delta$, $I_\varphi(x) < \delta$, $|\vartheta| < \delta$ it follows $\|\vartheta U_n(x_E)\| \leqslant \varepsilon$ for $n = 1, 2, \ldots$, (β) in the case of infinite (a, b) there exist, in addition to it, for every $\varepsilon > 0$ positive integer m and $\delta > 0$ such that for arbitrary measurable set E $\|\vartheta U_n(x_{(-\infty, c)\cap E})\| < \varepsilon$ if $c \leqslant -m$ (if $a = -\infty$), $\|\vartheta U_n(x_{(d, \infty)\cap E})\| < \varepsilon$ for $d \geqslant m$ (if $b = \infty$), with $n = 1, 2, \ldots$, if only $I_\varphi(x) < \delta$, $|\vartheta| \leqslant \delta$.

1.3. U is φ-linear if and only if

(α) U is absolutely continuous,

(β) U as operation in $M(a, b)$ is continuous in the case of finite (a, b), in the case of infinite (a, b) U is continuous in $M_0(a, b)$ in the sense of *γ-convergency*.

Necessity. In the case of finite (a, b) from $\sup_{(a, b)}^{*} |x_n(t)| \to 0$, and in the case of infinite (a, b) from *γ-convergence* of x_n to 0, it follows $I_\varphi(x_n) \to 0$, whence (β). If $|E_n| \to 0$ then $I_\varphi(x_{E_n}) \to 0$ and for infinite (a, b) we get also $I_\varphi(x_{(-\infty, -n)\cap E}) \to 0$, $I_\varphi(x_{(n, \infty)\cap E}) \to 0$ with $n \to \infty$, whence (α).

Sufficiency. Let $A_\delta = \{t: |x(t)| > \delta, t \in (a, b)\}$, $B_\delta = (a, b) - A_\delta$. If $I_\varphi(x) < \delta^2$ then $|A_\delta| < \delta$, whence applying 1.1 to $U_n = U$ we obtain $\|U(x_{A_\delta})\| < \varepsilon$ for sufficiently small δ. For infinite (a, b) there exists, in addition to it, an m such that $\|U(x_{(-\infty, -m)\cap E})\| < \varepsilon$, $\|U(x_{(m, \infty)\cap E})\| < \varepsilon$. We have $|x_{B_\delta}(t)| \leqslant \delta$; applying ($\beta$) we obtain for finite (a, b) $\|U(x_{B_\delta})\| < \varepsilon$ for sufficiently small δ. If (a, b) is infinite then (β) implies $\|U(x_{(-m, m)\cap B_\delta})\| < \varepsilon$ for sufficiently small δ, i.e. $\|U(x_{B_\delta})\| = \|U(x_{(-\infty, -m)\cap B_\delta})\| + \|U(x_{(-m, m)\cap B_\delta})\| + \|U(x_{(m, \infty)\cap B_\delta})\| < 3\varepsilon$. Finally $\|U(x)\| = \|U(x_{A_\delta}) + U(x_{B_\delta})\| \leqslant 2\varepsilon$ (or 4ε) if $I_\varphi(x) < \delta^2$.

1.4. If U_n are φ-linear and 1.1 (*) holds then U_n are *equi-φ-continuous* at 0.

Let A_δ, B_δ be sets as in the proof of 1.3. For $I_\varphi(x) \leqslant \delta^2$ we have $|A_\delta| < \delta$, whence by 1.1, (α) $\|U_n(x_{A_\delta})\| < \varepsilon$ for $n = 1, 2, \ldots$ Suppose that (a, b) is finite. Since U_n are continuous in $M(a, b)$ with respect to norm $\sup^*_{(a,b)} |x(t)|$, it follows that $U_n(x_n) \to 0$ if $\sup^*_{(a,b)} |x_n(t)| \to 0$. Hence $\|U_n(x_{B_\delta})\| < \varepsilon$ and also $\|U_n(x)\| < 2\varepsilon$ $(n = 1, 2, \ldots)$ for sufficiently small $\delta > 0$. In the case of infinite (a, b) the proof is similar and uses 1.1, (β), 1.3, (β).

1.4 gives

1.5. If U_n are φ-linear and $U_n(x)$ is bounded for $x \in L^{*\varphi}(a, b)$ then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $I_\varphi(x) < \delta$ implies $\|\vartheta U_n(x)\| < \varepsilon$ for $n = 1, 2, \ldots$, $|\vartheta| \leqslant \delta$.

1.6. Let U_n be absolutely continuous operations taking values from a B-normed space. If $U_n(x)$ are bounded for $x \in L^{*\varphi}(a, b)$ then there exists a constant $k > 0$ such that

$$(*) \qquad \|U_n(x)\| \leqslant k \sup(I_\varphi(x), 1) \quad \text{for} \quad n = 1, 2, \ldots$$

It follows by 1.2 that $\|U_n(x_E)\| \leqslant \bar{k}$ for $n = 1, 2, \ldots$, $I_\varphi(x) < 1/s$, $|E| < (b-a)/s$, where s is a sufficiently large positive integer. Let (a, b) be finite; if $I_\varphi(x) \leqslant 1$ then there exist at most s^2 disjoint sets E_i in (a, b) such that $I_\varphi(x_{E_i}) \leqslant 1/s$, $|E_i| \leqslant (b-a)/s$, and in view of $x = x_{E_1} + \ldots + x_{E_{s^2}}$ we have $\|U_n(x)\| \leqslant \bar{k}s^2$. Let $I_\varphi(x) = r + l$, where r is an integer $\geqslant 1$, $0 \leqslant l < 1$. We choose disjoint sets E_i, $i = 1, 2, \ldots, r$ and E after such a manner that $I_\varphi(x_{E_i}) = 1$, $I_\varphi(x_E) = l$, whence by $x = x_{E_1} + \ldots + x_{E_r} + x_E$ we get $\|U(x)\| \leqslant \bar{k}s^2(r+1) \leqslant \bar{k}s^2(I_\varphi(x)+1) \leqslant k \sup(I_\varphi(x), 1)$, where $k = 2\bar{k}s^2$. In the case of infinite (a, b) we appeal to 1.2, (β) and then argue similarly.

1.7. In order that operation U taking values from a B-normed space be φ-linear it is necessary and sufficient that it be absolutely continuous. Necessity follows from 1.3 (α); for sufficiency let us observe that 1.6 with $U_n = U$ implies continuity of U in $M(a, b)$ with respect to $\sup^*_{(a,b)} |x(t)|$ in the case of finite (a, b) and in the case infinite (a, b) it does imply *γ-continuity* in $M_0(a, b)$, whence 1.3, (β).

Note that one may replace in theorems 1.6, 1.7 B-normed domain of operations by a r-homogenous-normed one.

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Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

On Integral Representability of Linear Functionals over the Space of φ-integrable Functions

by
W. ORLICZ

Presented on July 6, 1960

1. In this note we use the notation and definitions introduced in [2], [4] and will occupy ourselves with the integral representation of φ-linear functionals in the space $L^{*\varphi}(a, b)$.

It is known that the condition $\lim_{u\to\infty} \varphi(u)/u > 0$ is necessary and sufficient for existence of a non-trivial φ-linear functional in $L^{*\varphi}(a, b)$ ([3], [5]). In the following we shall assume a somewhat stronger condition $\lim_{u\to\infty} \varphi(u)/u = \infty$ *which will not be formulated in assumptions*. On this condition we shall give an integral representation of φ-linear functionals.

1.1 Let Ξ denote the set of linear functionals over space $[L^{*\varphi}(a, b), \|\ \|_\varphi]$, further Ξ_φ the set of φ-linear functionals over $L^{*\varphi}(a, b)$. It is obvious that $\Xi_\varphi \subset \Xi$.

1.2. If $\xi \in \Xi$ then $\sup |\xi(x)| < \infty$ where sup is taken in $K_\varphi = \{x: I_\varphi(x) \leqslant 1\}$.

Choose an integer s so that $\|x\|_\varphi < 1/s$ would imply $|\xi(x)| \leqslant 1$. Suppose $I_\varphi(x) \leqslant 1$; choose s sets A_i, $A_i \cap A_j = 0$ $(i \neq j)$ so that $I_\varphi\left(\frac{s^{-1} x_{E_i}}{s^{-1}}\right) \leqslant \frac{1}{s}$. We have $\|s^{-1} x_{E_i}\|_\varphi < 1/s$, whence $s^{-1}|\xi(x_{E_i})| \leqslant 1$, i.e. $|\xi(x)| \leqslant s^2$.

1.3. Let us define a norm in Ξ by $\|\xi\| = \sup_{x \in K_\varphi} |\xi(x)|$; Ξ is a Banach space in this norm and Ξ_φ a closed linear subspace.

We omit the easy proof in which one should note that $\|x\|_\varphi < 1$ implies $x \in K_\varphi$.

1.31. We have the inequality $|\xi(x)| \leqslant \|\xi\|\,(I_\varphi(x)+1)$.

This follows from the definition of $\|\xi\|$ and takes a similar argument to that used in [4], 1.6.

1.4. Replacing, if necessary, φ by a function equivalent to it we can assume freely that $\varphi(u)/u \to 0$ if $u \to 0$.

Then the function

$$\psi(v) = \sup_{u>0} (uv - \varphi(u)) \quad \text{for} \quad v \geqslant 0$$

is referred to as *complementary function* to the function φ. It is easily seen that ψ is a convex φ-function ([1]) for which $\psi(v)/v \to \infty$ as $v \to \infty$. $u, v \geqslant 0$ being arbitrary, Young's generalized inequality holds. (Young's inequality follows if we assume φ to be convex)

$$(+) \qquad uv \leqslant \varphi(u)+\psi(v).$$

For every $v \geqslant 0$ there exists a u_v such that $u_v v = \varphi(u_v)+\psi(v)$ (though sign $=$ $=$ in $(+)$ must not occur for a given u). For $0 \leqslant v \leqslant v_0$ the set of corresponding u_v is bounded. Hence it follows easily

1.41. If y is measurable in (a, b) then there exists a measurable function x such that

$$x(t)\, y(t) = \varphi(x(t))+\psi(y(t)) \quad \text{for} \quad t \in (a, b),$$

where x is bounded in (a, b) if y is.

1.5. The general form of φ-linear functional in $L^{*\varphi}(a, b)$ is

$$(*) \qquad \xi(x) = \int_a^b x(t)\, y(t)\, dt,$$

where $y \in L^{*\psi}(a, b)$, where ψ is the function complementary to φ, more exactly we have $I_\psi(y/\|\xi\|) \leqslant 1$.

The integral $(*)$ will be denoted for brevity by $I(xy)$. Let $y \in L^{*\psi}(a, b)$, i.e. $I_\psi(\lambda y) < \infty$ with a certain $\lambda > 0$. Young's generalized inequality gives

$$\mu\,|x(t)| \cdot \lambda|\, y(t)| \leqslant \varphi(\mu|x(t)|)+\psi(\lambda|y(t)|),$$

whence the existence of integral $(*)$ for $x \in L^{*\varphi}(a, b)$ and its absolute continuity as functional in $L^{*\varphi}(a, b)$. ξ is therefore φ-linear by [4], 1.7. For the proof of necessity suppose that (a, b) is finite and observe that the set-function $\Phi(E) = \xi(\chi_E)$ (χ_E being the characteristic function of E) is additive and absolutely continuous function of set in (a, b) since $|E_n| \to 0$ implies $I_\varphi(\chi_{E_n}) \to 0$, whence $\Phi(E) = I(\chi_E y)$, where $y \in L^1(a, b)$. Hence, $\xi(x) = I(xy)$ for an arbitrary simple function. Approximating a measurable bounded function in (a, b) uniformly by simple functions we get $(*)$ for bounded x. Let y_n^0 denote the trunrated functions for $|y(t)|$; by 1.41 there exists a measurable bounded function $x_n(t)$ such that

$$|x_n(t)|\,\frac{y_n^0(t)}{\|\xi\|} = \varphi(x_n(t))+\psi\left(\frac{y_n^0(t)}{\|\xi\|}\right),$$

and consequently $I(x_n y_n^0/\|\xi\|) = I_\varphi(x_n)+I_\psi(y_n^0/\|\xi\|)$. But $I(x_n y_n^0) \leqslant I(|x_n|\,|y|)$, whence if $I_\varphi(x_n) \leqslant 1$ then $I(x_n y_n^0/\|\xi\|) \leqslant 1$ and further $I_\psi(y_n^0/\|\xi\|) \leqslant 1$; if $I_\varphi(x_n) > 1$ then by 1.31 $I(x_n y_n^0/\|\xi\|) \leqslant I_\varphi(x_n)+1$, what gives $I_\psi(y_n^0/\|\xi\|) \leqslant 1$. By Fatou's lemma we obtain $I_\psi(y/\|\xi\|) \leqslant 1$. We have proved $(*)$ for bounded functions and also that the right-hand side is a φ-linear functional, whence, as one can approximate in the sense of φ-convergency by bounded functions, arbitrary $x \in L^{*\varphi}(a, b)$, $(*)$ holds in all the space. The case of infinite (a, b) can be easily reduced to that of finite interval.

Similarly we show:

1.6. Linear functional in $[L^{*\varphi}(a, b), \| \|]$ is of form 1.5, (*) in subspace $[M^{\varphi}(a, b), \| \|_{\varphi}]$ resp. $[M_0^{\varphi}(a, b), \| \|_{\varphi}]$ if (a, b) is finite resp. infinite.

1.7. There exists a complementary function to function ψ as defined in 1.4. Let us write $\bar{\varphi}$ for it. ψ being convex there exists for every $u \geqslant 0$ a v_u with $\bar{\varphi}(u)+\psi(v_u) = uv_u \leqslant \varphi(u)+\psi(v_u)$. As also $\psi(v)+\varphi(u_v) = u_v v \leqslant \bar{\varphi}(u_v)+\psi(v)$, we have (+) $\bar{\varphi}(u) \leqslant \varphi(u)$ for $u \geqslant 0$ and (++) $\varphi(u_v) = \bar{\varphi}(u_v)$ for every u_v for which 1.4. (+) holds with the sign of equality. Suppose φ to satisfy

$$\text{(o)} \qquad \lim_{u\to\infty} \frac{\varphi(\lambda_0 u)}{\varphi(u)} = \infty \quad \text{with a certain} \quad \lambda_0 > 1.$$

From (+), (++) and (o) it follows $\lim_{v\to\infty} \frac{\varphi(u_v)}{\bar{\varphi}(u_v \lambda_0^{-1})} \geqslant \lim_{v\to\infty} \frac{\varphi(u_v)}{\varphi(u_v \lambda_0^{-1})} = \lim_{v\to\infty} \frac{\varphi(u_v)}{\varphi(u_v \lambda_0^{-1})} = \infty$. The last inequality shows that $\bar{\varphi}$ does not satisfy (Δ_2) for large u (as $v \to \infty$ then $u_v \to \infty$) and with a certain $\lambda > 1$ it does not hold $\varphi(\lambda u) = 0(\varphi(u))$ for large u. By this and (+) we get $L^{*\varphi}(a, b) \subset L^{*\bar{\varphi}}(a, b)$ and, there exists an element $x_0 \in L^{*\varphi}(a, b)$ such that $x_0 \in L^{*\bar{\varphi}}(a, b) - M^{\bar{\varphi}}(a, b)$. There exists a functional, vanishing at $M^{\bar{\varphi}}(a, b)$ and $\neq 0$ for x_0, which is linear in $[L^{*\bar{\varphi}}(a, b), \| \|_{\varphi}^0]$, $\| \|_{\varphi}^0$ being the homogeneous norm equivalent with the norm generated by $\bar{\varphi}$ (compare [2]). ξ is a linear functional in $[L^{*\varphi}(a, b), \| \|_{\varphi}]$ since $\|x_n\|_{\varphi} \to 0$ implies $\|x_n\|_{\varphi}^0 \to 0$ and 1.5, (*) does not hold. In view of the familiar theorem we obtain:

If $\lim_{u\to\infty} \frac{\varphi(\lambda_0 u)}{\varphi(u)} < \infty$ with a certain $\lambda_0 > 1$ then the linear functional in $[L^{*\varphi}(a, b), \| \|_{\varphi}]$ is of form 1.5, (*); if (o) is satisfied there exist linear functionals over $[L^{*\varphi}(a, b), \| \|_{\varphi}]$ which have no integral representation.

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POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

Proof of Mostowski's Conjecture

by

R. O. GANDY

Presented by A. MOSTOWSKI on June 23, 1960

Mostowski has conjectured that, given any arithmetic formula $A(\alpha, a)$, we can find a corresponding arithmetic formula $B(\beta, a)$ such that

$$(1) \qquad (\alpha)\, A(\alpha, a) \equiv (B\beta_{\text{hyp}})\ (E\beta, a).$$

In the above statement α and β are the only function variables which occur respectively in the formulae A and B. The converse problem of determining $A(\alpha, a)$, when $B(\beta, a)$ is given, was solved by Kleene*).

A proof of Mostowski's conjecture has been given in [5]. We give an alternative proof. We shall in fact construct a recursive functional $R(\alpha, x, a)$ such that

$$(2) \qquad a \in O \equiv (E\beta')\,(x)\, R(\beta' x, a);$$

here and in future we use primed Greek letters for variables which range only over hyperarithmetic functions.

Our proof of the conjecture is based on the fact that there are recursive linear orderings which contain well-ordered segments of order type ω_1; we start by proving that this is the case. Let $R(x, y)$ be a recursive linear ordering of natural numbers; we understand by this that R satisfies the usual axioms when $R(x, y)$ is interpreted as meaning that x strictly precedes y. Let F_R be the field of R; we define *the initial segment* of R (denoted by 'I_R') to be the greatest initial segment of F_R which is well-ordered by R; thus

$$y \in I_R \equiv y \in F_R \,\&\, \{(\lambda x R(x, y)) \text{ is well-ordered by } R\}.$$

Thus, we define *the initial ordinal* of R to be the order type of I_R.

THEOREM 1. *Let R be a recursive linear ordering which is not a well-ordering; then, if the initial ordinal of R is less than ω_1, there is an infinite hyperarithmetical sequence which is strictly decreasing on R.*

Let R satisfy the conditions of the theorem; then there is a recursive well-ordering $S(u, v)$ which is isomorphic with I_R. We first show that I_R is hyperarithmetic. This

*) See [2] and [4]; in the latter paper Mostowski's conjecture is mentioned as being an open problem.

is trivially true if I_R is empty. Suppose that I_R is not empty, and consider the property $M(\alpha, R, S)$ defined by

$$M(\alpha, R, S) \equiv (v)\{\alpha(v) \in F_R \,\&\, (x)[R(x, \alpha(v)) \equiv (Eu)\{S(u, v) \,\&\, x = \alpha(u)\}]\}.$$

$M(\alpha, R, S)$ holds just in case α is the unique one-to-one order preserving map of the field of S onto a segment namely I_R of F_R which also maps the non-field of S onto the first element of F_R. Since R and S are recursive, F_R is arithmetic, and $M(\alpha, R, S)$ is arithmetic in α. Hence, I_R is hyperarithmetic, for we have

$$x \in I_R \equiv (E\alpha)(Eu)[M(\alpha, R, S) \,\&\, x = \alpha(u)],$$

$$\equiv (\alpha)(Eu)[M(\alpha, R, S) \supset x = \alpha(u)].$$

Now, since R is not a well-ordering, there must be an x such that $x \in F_R$ & $x \notin I_R$; and for each such x there must be a y such that $R(y, x)$ & $y \notin I_R$, for otherwise $I_R \cup \{x\}$ would be a well-ordered initial segment of F_R. Hence the function β defined by

$$\beta(0) \equiv \mu x[x \in F_R \,\&\, x \notin I_R],$$

$$\beta(n+1) \equiv \mu y[R(y, \beta(n)) \,\&\, y \notin I_R],$$

gives an infinite sequence which is strictly decreasing on the ordering R. But β is recursive in F_R and I_R; it is therefore hyperarithmetic and the theorem is proved.

COROLLARY. *ω_1 is the initial ordinal of any recursive linear ordering which is not a well-ordering but which is a well-ordering with respect to all hyperarithmetic sequences.*

For, by the theorem, the initial ordinal of such an ordering cannot be less than ω_1; and if it were greater there would be a w such that the segment $\lambda x R(x, w)$ was of order type ω_1.

Now let $K(\alpha)$ be a predicate of functions which is satisfied by some function but by no hyperarithmetic function, and which is of the form $(x)J(\bar{\alpha}(x))$ with recursive J.*) Let S_1^J be the set of sequence numbers which are not past secured with respect to J; and let $u \prec v$ be the relation between sequence numbers which corresponds to the lexicographic ordering of sequences. We set

$$R(u, v) \equiv u, v \in S_1^J \,\&\, u \prec v.$$

Then R satisfies the conditions of the corollary*), and so its initial segment has order type ω_1. And we can determine this initial segment rather easily; in fact

$$u \in I_R \equiv u \in S_1^J \,\&\, (\beta)[K(\beta) \supset u \not\succeq \beta(lh(u))]. \tag{3}$$

For, suppose u satisfies the R.H.S. of (3); then u is securable, and so is any $v \prec u$; hence $\lambda v R(v, u)$ is well-ordered by R, and $u \in I_R$. Conversely, suppose that u does not satisfy the R.H.S.; if $u \notin S_1^J$ then $u \notin F_R$ and *a fortiori* $u \notin I_R$; while if $u \in S_1^J$

*) The existence of such predicates has been established by Kleene (Theorem XXVI in [2]); he has also discussed the way in which they determine orderings which are well-orderings with respect to hyperarithmetic sequences (see [1], pp. 421—2).

and $\bar{\beta}(lh(u)) \prec u$ for some β which satisfies $K(\beta)$, then $\lambda n \beta(lh(u)+n)$ is an infinite descending sequence of elements of F_R, and so $u \notin I_R$.

Remarks. 1. From (3) we see that there is Π^1_1 ordering of integers of order type ω_1.*)

2. Suppose we extend every finite sequence (with sequence number u, say) to an ideal infinite sequence u^* by putting

$$u^*(n) = \omega \qquad \text{if } n \geqslant lh(u),$$
$$= (u)_n \cdot 1 \quad \text{if } n < lh(u),$$

and then order all (ordinary and ideal) infinite sequences lexicographically, with the convention $(x)(x < \omega)$. This induces an ordering between all (finite and infinite) sequences which for finite sequences agrees with the ordering introduced by Kleene; we write '$u \prec \beta$', if u^* comes after β lexicographically. Now let B be the greatest lower bound of all sequences β which satisfy $K(\beta)$; then B is an infinite sequence which itself satisfies $K(\beta)$**) and we can rewrite (3):

$$u \in I_R \equiv u \in S_1^J \;\&\; u \prec B.$$

From (3) it follows that I_R is recursive in O, and O is certainly hyperarithmetic in I_R (and in B); but I think it possible that, for suitable K, $\boldsymbol{O}$ is not recursive in H_a^B (where $a \in \boldsymbol{O}^B$) if $|a|^B < \omega_1$.

THEOREM 2. *The predicate $a \in \boldsymbol{O}$ can be expressed in the form $(E\beta')(x)\, R(\beta', x, a)$.*

To prove the theorem we first construct a recursive order-preserving map of I_R into $\boldsymbol{O}$. Suppose that Φ satisfies

$$\Phi(u) = 1 \qquad \text{if } u \text{ is the first element of } I_R;$$
$$= 2^{\Phi(P(u))} \quad \text{if } u \text{ is secured;}$$
$$= 3.5^{f_u} \qquad \text{if } u \text{ is not secured;}$$

where $P(u)$ is the immediate predecessor of u in I_R, and f_u represents $\Phi(u*2^n)$ as a function of n_O. Then Φ is an order-preserving map of I_R into O. Also P is recursive***) and f_u is uniformly recursive in Φ, u; hence we can find a recursive Φ which satisfies the above conditions.

By Theorem 1, for any $a \in O$ there is a $u \in I_R$ such that $|\Phi(u)| = |a|$. Therefore (see corollary 1 to theorem 4 of Spector [2])

$$a \in O = (Eu)\,[u \in I_R \;\&\; R_1(H_{\Phi(u)**}, a, \Phi(u))].$$

Thus the theorem will be proved, if we can express $u \in I_R$ in the form $(E\alpha')(x)\, R(\alpha', x, u)$.

*) The existence of such orderings has also been recently established by Feferman.

**) B is in fact the function which Kleene constructs by a recursion in one-function-quantifier predicates in section 5.5 of [3].

***) If u is the number of the sequence $(x_1, x_2, \ldots, x_r, 0, \ldots, 0)$, where $x_r > 0$, then $P(u)$ is the number of the sequence $(x_1, x_2, \ldots, x_r - 1, 0, \ldots, 0)$.

Let Ψ be a variable for predicates of two numerical arguments. Let $C(\Psi, a, x)$ be an arithmetic formula which expresses the definition of $H_a(x)$ in terms of the predicate $\lambda by\,\{b < O_a \,\&\, H\ (y)\}$. Set

$$D(\Psi, u) \equiv u \in S_1^J \ \&\ (v)\,\{v \in S_1^J \ \&\ v \prec u \cdot \supset (x)\,[\Psi(\Phi(v), x) \equiv C(\Psi, \Phi(u), x)]\}.$$

If $u \in I_R$, then $D(\Psi, u)$ is satisfied by any predicate Ψ such that $\lambda x \Psi(b, x) = H_b$ for all $b \leqslant_O \Phi(u)$; in particular $D(\Psi, u)$ is satisfied by the hyperarithmetic predicate $\lambda bx\,\{b \leqslant_O \Phi(u) \ \&\ H_b(x)\}$. But if $u \notin I_R$, then u, as an element of F_R, comes after every element of I_R; so if $D(\Psi, u)$ holds, $\lambda x \Psi(\Phi(v), x) = H_{\Phi(v)}$ for every v in I_R. But then Ψ cannot be hyperarithmetic. Thus

$$u \in I_R \equiv (E\Psi)_{\mathrm{hyp}}\, D(\Psi, u).$$

Finally, since D is arithmetic, it can be expressed in the form $(E\gamma)$ (γ is arithmetic in Ψ & $(x)\, R_1(\Psi, x, u)$) with recursive R_1. So, replacing Ψ by its representing function and amalgamating like quantifiers, we obtain expressions for $u \in I_R$ and $a \in O$ which have the requisite form.

An examination of the arguments used so far shows that they can all be relativised to a finite string of arbitrary functions γ; in particular

$$a \in O^\gamma \equiv (E\beta)_{\mathrm{hyp}(\gamma)}\,(x)\, P^\gamma(\bar{\beta}(x), a), \tag{4}$$

where the suffix indicates that the variable β ranges over functions hyperarithmetic in functions of the string γ. Similarly Kleene's proof (see footnote*) on p. 571) can be relativised; but it also remains valid without relativisation when free function variables are introduced. The following theorem summarises these extensions of Kleene's result and the main result of this paper.

THEOREM 3. *Let $A(\alpha, \gamma, z)$ be an arithmetic predicate of the string z of number variables relative to α and the string γ of function variables. Then we can find recursive predicates $P_i(u, v, z)$ $(i = 1, 2, 3)$ such that*

(i) $$(E\alpha)_{\mathrm{hyp}(\gamma)}\,(x)\, A(\alpha(x), \dot{\gamma}(x)\, z,) \equiv (\beta)\,(Ex)\, P_1(\bar{\beta}(x), \bar{\gamma}(x), z),$$

(ii) $$(E\alpha)_{\mathrm{hyp}}\,(x)\, A(\alpha(x), \dot{\gamma}(x), z) \equiv (\beta)\,(Ex)\, P_2(\bar{\beta}(x), \dot{\gamma}(x), z),$$

(iii) $$(\alpha)\,(Ex)\, A(\alpha(x), \bar{\gamma}(x), z) \equiv (E\beta)_{\mathrm{hyp}(\gamma)}\,(x)\, P_3(\bar{\beta}(x), \bar{\gamma}(x), z).$$

Furthermore, the Gödel numbers of the P_i can be determined by primitive recursions from the Gödel number of A.

To prove (iii) we use (4) and the fact that O^γ is a complete predicate for predicates of the form of the L.H.S. of (iii). The last sentence of the theorem follows from the fact that at each stage of the reduction processes by which (i)—(iii) are proved one can write out all the relevant formulae explicitly.

In general the device of Kleene ([3], par. 3.4) for commuting number and function quantifiers of opposite kinds cannot be applied when the range of the function quantifiers is restricted. But one can sometimes apply (i) and (iii) of Theorem 3 alternately to secure the same effect. In particular the following theorem is easily proved.

THEOREM 4. *Let $F(\gamma, z)$ be a formula in prenex normal form with the strings γ, z as its only free variables, and recursive functions and predicates as its only non-logical constants. Further, let its bound function variables be either* (a) *all universally, or* (b) *all existentially quantified, and let them be restricted to range only over functions hyperarithmetic in the variables of γ. Then we can find recursive $P(u, v, z)$, $Q(u, v, z)$ such that:*

$$F(\gamma, z) = (\beta)_{\mathrm{hyp}(\gamma)}(Ex)\, P(\beta(x), \bar{\gamma}(x), z) \text{ in case (a),}$$

$$F(\gamma, z) = (E\beta)_{\mathrm{hyp}(\gamma)}(x)\, Q(\bar{\beta}(x), \bar{\gamma}(x), z) \text{ in case (b).}$$

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

Set Existence

by

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Presented by A. MOSTOWSKI on August 18, 1960

1. We consider sets of axioms $\mathcal{A}$ in the notation of the finite theory of types (using, for convenience, only set variables and no function variables). The axioms contain the relation symbol S.

Ω-models of $\mathcal{A}$ [1]. The individuals range over the set N of natural numbers and $S(x, y)$ is interpreted as the successor relation. The variables of higher type range over such subsets of the maximal (or: principal) model as satisfy $\mathcal{A}$.

The sets which belong to the common part of all ω-models of $\mathcal{A}$ are called *assured by A* (under the ω-model interpretation). It is clear that elements of an assured set are assured. A weaker notion than *belonging to* a model is also useful for sets of higher type, and, generally, in the case when the ranges of the given type are not the same in all models.

Inductive definition of: *The set Y represents X in the model M.* (i) If $X \subset N$, $Y = X$; (ii) if $\varkappa^\alpha$ is the collection of all those sets of type α which are represented in each ω-model of $\mathcal{A}$ (by some set) and $X \subset \varkappa^\alpha$, then Y represents X in M provided Y is in M and

$$(X')_{\varkappa^\alpha}(Y')(Y'')[(Y' \text{ represents } X' \ \&\ Y'' \text{ represents } X') \to (Y' \in Y \leftrightarrow Y'' \in Y)]$$

$$(X')_{\varkappa^\alpha}[X' \in X \leftrightarrow (E\,Y')(Y' \text{ represents } X' \ \&\ Y' \in Y)]\ *).$$

It is plausible, and proved in Theorem 2 below, that there are sets of higher type which are represented in each ω-model of $\mathcal{A}$ without being assured by $\mathcal{A}$.

General models. To avoid complications in the 'representations' of natural numbers, we consider only axiom systems which include

$$(x)(Ey)\,S(x, y), \quad (E!\,y)(x) \neg S(x, y), \quad (x)(y)(z)\{[S(x, y) \,\&\, S(x, z)] \to y = z\}.$$

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*) This is the model theoretic counterpart of the (syntactic) notion of *definite term* in [3]. par. 4, corresponding to the "extensionality" condition (E_2). The condition on a representing set Y corresponding to (E_1) is: $(X')_{\varkappa^\alpha}(Y')(Y'')[(Y' \text{ represents } X' \ \&\ Y'' \text{ represents } X' \ \&\ Y' \in Y \ \&\ Y'' \in Y) \to Y' = Y'']$.

Then any model M contains objects $\gamma_0^M, \gamma_1^M, \ldots$ (not necessarily distinct) such that γ_0^M satisfies $(x) \neg S(x_0, x)$ in M, $\gamma_i^M, \gamma_{i+1}^M$ satisfy $S(x_i, x_{i+1})$ for $i = 0, 1, \ldots$. The set N_M of γ_i^M is called the set of natural numbers in M.*)

For a set X of natural numbers we say (a) X *belongs* to M, if there is a Y of M such that (i) $Y \subset N_M$, (ii) $X = \{n \; \gamma_n \in Y\}$, (b) as usual, Y *represents* X if (ii) holds but not necessarily (i). The notions of *assured* and *representable* set of arbitrary finite type are then introduced as above.

2. By a trivial extension of [1] to finite types, a subset of N is hyperarithmetic (HA), if it is defined by the same formula in our notation in each ω-model of $\mathcal{A}$ and $\mathcal{A}$ is of type Π_1^1. Analogous results hold for sets X such that there exists a formula Φ_X defining Y_X^M in the model M, where, in each model M of $\mathcal{A}$, Y_X^M represents X; for general models, see [1]; for subsets of higher type, see [3], and for sets of sets in ω-models, see Proposition D of [3]. These results leave open the question whether every subset of N which is assured by a Π_1^1 set of axioms (for ω-models), is HA; and related questions for higher types and for sets which are representable in all ω-models, resp. all general models. These questions are treated in the present note.

An interesting corollary to Theorem 1 is that the common part of all ω-models of full second order arithmetic (with the comprehension schema, as in [1]) is not a model for it, because the set of HA sets does not satisfy the comprehension schema.

Remark. The referee has pointed out that Theorem 3 is a special case of Theorem 1 of the abstract [4]; further that Theorem 1 can be obtained by the methods indicated in [4]; and, lastly, that a refinement of these methods shows: if a set of natural numbers is assured by a Π_1^1 set $\mathcal{A}$ (in ω-models) then it is *strongly representable* in $\mathcal{A}$ in the sense of [1], and if represented in all general models of a Σ_1^0 set $\mathcal{B}$ in the sense above, then it is *strongly representable* in $\mathcal{B}$. These are improvements of Theorems 1 and 3 since there are $\mathcal{A}$ in which not all HA sets (and $\mathcal{B}$ in which not all recursive sets) are strongly representable for ω-models (general models). However, the lemma which is used in our proofs of Theorems 1 and 3 is of independent interest since it constitutes a new basic result for predicates $(x) \; R[\alpha(x)]$, and $(x)\{\alpha(x) \leqslant 1 \;\&\; R[\overline{\alpha}(x)]\}$ with recursive R.

3. Lemma. *If R_0 is recursive and $q_1, q_2, \ldots$ is a sequence of functions such that either (a) each q_i is not* HA *or (b) for each q_i, $(x)\, q_i(x) \leqslant 1$ and q_i is not recursive then $(E\alpha)(x) \; R[\alpha(x)] \;\&\; (E\alpha^*)\{(x) \; R[\alpha^*(x)] \;\&\; (i) \; (q_i$ is not primitive recursive in $\alpha^*)\}$ where, in case (a) R denotes R_0, in case (b) R denotes $\alpha(x) \leqslant 1 \;\&\; R_0$.*

Proof. *Let $\Phi_n(\alpha, y)$ be an enumeration of all primitive recursive functionals, and let $\langle n_1, n_2 \rangle$ be the n^{th} pair in some standard enumeration of pairs of natural numbers.*

Define the sequence a_n by induction as follows (where α^* will be taken as: $\alpha^*(n) = (a_{n+1})_n$):

*) Since the ranges of the individual variables in different (general) models can be different some convention is needed to match them up.

$$a_0 = 1$$

$$a_{n+1} = (\mu a)\,[(Ex)_{\leq 1}\,(a = a_n * x)\ \&\ (E\alpha)\,\{a = \alpha\,[lh\,(a)]\ \&\ (x)\,R\,[\bar{\alpha}\,(x)]\ \&$$
$$(Ey)\,([q_{n_1}\,(y) \neq \Phi_{n_2}\,(\alpha, y)]\ \&\ (\beta)\,\{\beta\,[lh\,(a)] = a \to \Phi_{n_2}\,(\beta, y) = \Phi_{n_2}\,(\alpha, y)\})].$$

For each n, the μ-operator is effective. Otherwise a condition of the form $(\alpha)\,\{A\,(\alpha) \to (y)\,[q_{n_1}\,(y) = \Phi_{n_2}\,(\alpha, y)]\}$ would be satisfied, and so

$$(x)\,(y)\,[q_{n_1}\,(x) = y \leftrightarrow (\alpha)\{A\,(\alpha) \to \Phi_{n_2}\,(\alpha, y) = x\}].$$

(a) This means that the graph of q_{n_1} is Π^1_1 and hence q_{n_1} is HA.

(b) Since $A\,(\alpha) \to \alpha\,(x) \leqslant 1$, by König's lemma the graph of q_{n_1} is Σ^0_1, and so q_{n_1} is recursive.

α^* defined above satisfies $R\,[\alpha^*\,(n)]$ for each n by the definition, and, for each n, $(Ey)\,[q_{n_1}\,(y) \neq \Phi_{n_2}\,(\alpha^*, y)]$ by the continuity of primitive recursive functionals. Since each pair of numbers $\langle s, t\rangle$ appears in the enumeration, no q_s is primitive recursive in α^*.

COROLLARY. (In terms of the notion of basis). *For any denumerable set F of functions none of which is* HA, *the set of functions in which no element of F is primitive recursive, form a basis for* $(x)\,R\,[\bar{\alpha}\,(x)]$ (*with the analogous result for* $(x)\,\{\alpha\,(x) \leqslant 1\ \&\ R\,[\bar{\alpha}\,(x)]\}$).

Remark. We shall use the lemma only for a single q. The proof evidently establishes a stronger result, in case (a) with *hyperarithmetic* in place of *primitive recursive*.

4. We consider first subsets of N for ω-models; N is the common range of lowest type in all the models considered, and so this case is special.

THEOREM 1. *If (the set of Gödel numbers of)* $\mathscr{A}$ *is* Π^1_1 *then the intersection of all* ω*-models of* $\mathscr{A}$ *contains only* HA *sets of natural numbers.*

It is sufficient to consider denumerable models of $\mathscr{A}$. We reserve a natural number q for each type other than the individuals, $q = 0$ for subsets of N. We describe the model by a property $P\,(\langle n, m\rangle)$ as follows, with $n = 2^q\,(2r+1)$. For $q = 0$, $\lambda mP\,(\langle n, m\rangle)$, $r = 0, 1, \ldots$, is an enumeration of the subsets of N in the model. If q' denotes the type of sets of type q, and $\lambda sP\,[2^q\,(2m+1), s\rangle]$ represents the set S_m (of type q), then $\lambda mP\,[\langle 2^{q'}\,(2r+1), m\rangle]$ represents $\{S_m : P\,[\langle 2^{q'}\,(2r+1), m\rangle]\}$; for $r = 0, 1, \ldots$, $\lambda mP\,[\langle 2^{q'}\,(2r+1), m\rangle]$ is an enumeration of all the sets of type q' of the model. The condition that P enumerates the sets of an ω-model of $\mathscr{A}$ is expressed by $(E\beta)\,(x)\,R_A(P, \beta, x)$, where β is a function (not set) variable and R_A is recursive in P and β. $(EP)\,(E\beta)\,(x)\,R_A\,(P, \beta, x)$ is contracted in the usual manner to $(E\alpha)\,(x)\,R\,(\alpha, x)$ such that, if α satisfies $(x)\,R\,(\alpha, x)$, then there are P (and β), primitive recursive in α, which satisfy $(x)\,R_A\,(P, \beta, x)$. Thus, each subset of N enumerated by P is primitive recursive in α. By the lemma, case (a), for each subset X of N which is not HA there is a model to which X does not belong. This proves Theorem 1.

Remark. While, of course, each set which is strongly representable in the sense of [1] is assured, Theorem 1 shows that weak representability is essentially a relation

between *formulas* and the class of *all* the models considered; for, if X is weakly, but not strongly representable, there exists a model to which X does not belong.

5. THEOREM 2. (i) *Suppose the set of axioms $\mathcal{A}$ contains the successor axioms of paragraph* 1, *the axioms for a total ordering* $\leqslant$,

$$(x)(y)(z)\{S(y,z) \to [x \leqslant z \leftrightarrow (x \leqslant y \vee x = y)]\}$$

and, for each n,

$$(x)(x_1)\ldots(x_n)\{[S(0,x_1)\ \&\ \ldots\ \&\ S(x_{n-1},x_n)] \to [x \leqslant x_n \leftrightarrow (x = 0 \vee x = x_1 \vee \ldots \vee x = x_n)]\}.$$

Then every set which is assured in all general models of $\mathcal{A}$ is hereditarily finite.

(ii) *Suppose $\mathcal{A}$ is a Π^1_1 set of axioms which includes all instances of the comprehension axiom* $(EX^1)(x)[x \in X^1 \leftrightarrow P(x)]$ *where $P(x)$ does not contain bound variables of higher type.*

Then, if a set of sets of natural numbers is assured in all ω-models of $\mathcal{A}$, all its elements are recursive in some H_n, $|n| < \omega_1$, where H_n is Kleene's HA *set associated with the ordinal notation n.*

Proof of (i). Add to $\mathcal{A}$, for each n, the axioms $(x_1)\ldots(x_n)\{[S(0,x_1)\ \&, \ldots, \&\ S(x_{n-1},x_n)] \to u \neq x_n\}$, or: $u \neq 0^{(n)}$ for short, where u is a constant which does not occur in $\mathcal{A}$, and, for $m = 0, 1, \ldots$

$$(1)\quad (X^1)\{(0^{(n)} \in X^1\ \&\ (x)(Ey)[x \in X^1 \to (x < y\ \&\ y \in X^1)]) \to (Ez)(u \leqslant z\ \&\ z \in X^1)\}.$$

If $\mathcal{A}$ is consistent, so is the extended set, and hence the latter has a model M. Suppose X^1_0 is assured by $\mathcal{A}$. Then X^1_0 occurs in M, and, since $X^1_0 \subset N_M$ and $u_M \notin N_M$, $(z)(u_M \leqslant z \to z \notin X^1_0)$ in M by our axioms for $\leqslant$. Substituting X^1_0 for X^1 in (1), $(x)(Ey)[x \in X^1_0 \to (x < y\ \&\ y \in X^1_0)]$ is false in N_M, i.e. X^1_0 is finite. — A similar argument applies to higher types.

Proof of (ii). Note first that the assumed form of the comprehension principle ensures that H_n is well-defined as the unique set which satisfies a suitable *arithmetic* condition $\mathfrak{H}(X^{1\prime}, n)$. There is a formula $P(n)$ with a single universal function quantifier and arithmetic scope which (a) defines (in the principal ω-model) a set P consisting of a unique ordinal notation for each recursive ordinal ordered by $<_O$ and (b) satisfies

$$(2)\qquad (x)(y)\{[P(x)\ \&\ P(y)] \to (x <_O y \vee y <_O x \vee x = y)\}$$

in each ω-model of the sets $\mathcal{A}$ considered *).

We add to $\mathcal{A}$ the Π^1_1 set of axioms: for each $\bar{n} \in O$,

(i) U^1 is not recursive in $H_{\bar{n}}$, where U^1 is a constant not occurring in $\mathcal{A}$,

(ii) $(X^2)[\{(EX^1)(X^1 \in X^2\ \&\ X^1 \text{ is recursive in } H_{\bar{n}})\ \&\ (X^1)(x)(EY^1)[\{X^1 \in X^2\ \&\ P(x)\ \&\ H_x \text{ recursive in } X^1\} \to (Y^1 \in X^2\ \&\ H_{2^x} \text{ recursive in } Y^1)]\ \&\ (e)[\{P(3.5^e)\ \&$

*) Such formulae $P(n)$ have been constructed (independently) in an unpublished work by S. Feferman and R. O. Gandy.

$(x)(EX^1)(H_{\{e\}(x)}$ recursive in $X^1 \& X^1 \in X^2)\} \to (EY^1)(H_{3.5^e}$ recursive in $Y^1 \& Y^1 \in X^2)]\} \to (EX^1)(U^1$ recursive in $X^1 \& X^1 \in X^2)]$.

If $\mathscr{A}$ has an ω-model so has the extended set of axioms. For, by [3], p. 120, a Π^1_1 set of axioms has an ω-model, if every HA subset has one; but every HA subset of (i) contains only $\bar{n}$ satisfying $|\bar{n}| < |n_0| < \omega_1$, and then (ii) has an ω-model if we take H_{n_0} for U^1.

Now consider a model M of the extended set of axioms and an assured set X^2_0. Since $\mathscr{A}$ is Π^1_1, by Theorem 1, all elements of X^2_0 are HA (this is our only use of this Π^1_1 character of $\mathscr{A}$). So, in M, we have

(3) $$\neg (EX^1)(U^1_M \text{ recursive in } X^1 \& X^1 \in X^2_0).$$

Now suppose

(4) For each $n \in O$, $(EX^1)(X^1 \in X^2_0 \ \&\ H_n$ recursive in $X^1)$.

(5) $(4) \to (X^1)(x)(EY^1)[\{X^1 \in X^2_0 \& P(x) \ \&\ H_x$ recursive in $X^1] \to (Y^1 \in X^2_0 \& H_{2x}$ recursive in $Y^1)]$.

Consider any X^1, x: if $X^1 \in X^2_0$ then X^1 is HA; if $P(x)$ holds in M, and H_x is recursive in a HA set H_m, $m \in O$, then $x \in P$, since by (2), $x <_O n \vee n <_O x$ and so, if $x \notin P$, $n <_O x$ for each $n \in O$, each H_n recursive in the sets of M defined by H_x and so H_x is not HA. Hence $2^x \in P$ too, and so $(EY^1)(Y^1 \in X^2_0 \& H_{2x}$ recursive in $Y^1)$ by (4). Similarly,

(6) $(4) \to (e)\{[P(3.5^e) \& (x)(EX^1)(H_{\{e\}(x)}$ recursive in $X^1 \ \&\ X^1 \in X^2_0)] \to (EY^1)(H_{3.5^e}$ recursive in $Y^1 \& Y^1 \in X^2_0)$.

By (3), (5) and (6), the assumption of (4) contradicts the axiom (ii) when $X^2 = X^2_0$. This refutes (4) and establishes Theorem 2 (ii).

The extension to higher types is in principle straightforward.

Note that Theorem 2 illustrates the analogy in [3] between finite sets in general models of *arbitrary* $\mathscr{A}$ and HA sets in ω-models of Π^1_1 sets $\mathscr{A}$.

6. Theorem 3. *If the subset X of N can be represented in all general models of a Σ^1_1 set of axioms $\mathscr{A}$ then X is recursive.*

Proof. Translate $\mathscr{A}$ into first order notation in the familiar way, e.g. by means of a set-membership relation. Eliminate function symbols by means of relation symbols and axioms expressing the functional character of these relations and put each axiom into Skolem's normal form

(7) $$(x_1 \ldots (x_n)(Ey_1) \ldots (Ey_m) A(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

where A is a quantifier-free formula whose non-logical constants are relation symbols.

If (7) has a general model with N as range of the individual variables, then (as in Gödel's proof of the completeness theorem and in contrast to ω-models) so has

(8) $$(x_1) \ldots (x_n) A[x_1, \ldots, x_n, \eta_1(x_1, \ldots, x_n), \ldots, \eta_m(x_1, \ldots, x_n)]$$

provided only the η_i are disparate functions; here $=$ is treated as an ordinary non-logical constant with the appropriate identity axioms and, in general, S is not

interpreted as the successor relation. In particular, in terms of the η_i we can express in advance one sequence of elements $\gamma_0, \gamma_1, \ldots$ which satisfy the conditions for the natural numbers in every general model of (8). If the η_i are chosen to be primitive recursive, so is the set $\gamma_0, \gamma_1, \ldots$ since the η_i are disparate and thus $\eta_i(x_1, \ldots, x_n) > x_j$, $1 \leqslant j \leqslant n$. By contracting the predicate symbols and individual variables of (8) in the usual manner, the existence of a general model of (8) is expressed by $(EP)(x)\, R_A(P, x)$, or equivalently $(E\alpha)(x)\{\alpha(x) \leqslant 1 \;\&\; R[\overline{\alpha}(x)]\}$. Note that the subsets of M which are *represented* in M (for our choice of $\gamma_0, \gamma_1, \ldots$) are primitive recursive in α. By the Lemma, case (b), for any non-recursive set X there is a model of (8) with identity treated as an ordinary relation in which X is not represented. By taking equivalence classes we get from this model one in which $=$ is interpreted by identity. Also, since models of (8) are models of (7), the theorem follows.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

A Necessary and Sufficient Condition for the Validity of the Weak Law of Large Numbers

by

A. EHRENFEUCHT and M. FISZ

Presented by E. MARCZEWSKI on June 30, 1960

Let $\{x_i\}$ $(i = 1, 2, \ldots)$ be a sequence of random variables. We say that the sequence $\{x_i\}$ obeys the weak law of large numbers, if there exists such a sequence of constants $\{a_n\}$ that for any $\varepsilon > 0$ the relation

$$\lim_{n\to\infty} P\left(\left|\frac{y_n}{n} - a_n\right| \geq \varepsilon\right) = 0 \tag{1}$$

holds, where $y_n = x_1 + \ldots + x_n$.

THEOREM. *Let $\{x_i\}$ $(i = 1, 2, \ldots)$ be a sequence of independent and equally distributed random variables, and let $\varphi(t)$ denote the common characteristic function of x_i. The sequence $\{x_i\}$ obeys the weak law of large numbers with $a_n = a$ $(n = 1, 2, \ldots)$ if and only if $\varphi(t)$ is derivable at $t = 0$ and $\varphi'(0) = ai$.*

Proof. Let the derivative of $\varphi(t)$ at $t = 0$ exist and let $\varphi'(0) = ai$. We then have

$$\varphi(t) = 1 + ait + o(t).$$

For the characteristic function $\varphi_n(t)$ of y_n/n we obtain:

$$\varphi_n(t) = [\varphi(t/n)]^n = [1 + ait/n + o(t/n)]^n,$$

$$\log \varphi_n(t) = ait + no(t/n).$$

Consequently, $\lim_{n\to\infty} \log \varphi_n(t) = ait$. This equality implies in virtue of the Lévy Theorem relation (1) with $a_n = a$.

Let now relation (1) hold with $a_n = a$. Denote by $\varphi_0(t)$ the characteristic function of the random variable $x_i - a$ $(i = 1, 2, \ldots)$. We then have

$$\lim_{n\to\infty} [\varphi_0(t/n)]^n = 1,$$

and, consequently, for any real t

$$\lim_{n\to\infty} \frac{\log\{1 - [1 - \varphi_0(t/n)]\}}{(t/n)} = 0. \tag{2}$$

Since $\varphi_0(0) = 1$, we obtain by the continuity of $\varphi_0(t)$ that $1 - \varphi_0(t/n)$ is arbitrarily small for any t and a sufficiently large n. Hence, relation (2) implies

$$\lim_{n\to\infty} \frac{1-\varphi_0(t/n)+o[1-\varphi_0(t/n)]}{(t/n)} = 0. \tag{3}$$

The last equality will hold only, if the equalities

$$\lim_{n\to\infty} \frac{1-\varphi_0(t/n)}{(t/n)} = \lim_{n\to\infty} \frac{\varphi_0(0)-\varphi_0(t/n)}{(t/n)} = 0 \tag{4}$$

hold for any real t.

We shall now show that if (4) holds for any real t, then $\varphi_0(t)$ is derivable at $t = 0$ and $\varphi_0'(0) = 0$, hence $\varphi'(0)$ exists and equals ai. This follows from the following

LEMMA. *If the function $\psi(t)$ satisfies the two conditions*:

(i) *$\psi(t)$ is continuous in the open intervals* $(-\infty, 0)$, $(0, +\infty)$, (ii) *for every* $t \neq 0$, $\lim_{n\to\infty} \psi(t/n) = 0$, *then*

$$\lim_{t\to 0} \psi(t) = 0. \tag{5}$$

Proof of the Lemma. Let the assumptions (i) and (ii) be satisfied and let relation (5) not hold. There exists then a sequence $\{t_n\}$ $(t_n \neq 0,\ n = 1, 2, \ldots)$ and a number $\delta > 0$ such that

$$\lim_{n\to\infty} t_n = 0 \tag{6}$$

and

$$|\psi(t_n)| > 2\delta \text{ for all } n. \tag{7}$$

Let $\langle t_n, t_n'\rangle$ be a closed interval, such that for any t from $\langle t_n, t_n'\rangle$:

$$|\psi(t)| > \delta. \tag{8}$$

The existence of such intervals for any n follows evidently from assumption (i) and relation (7).

Let us now define by induction two sequences of integers $\{n_k\}$ and $\{s_k\}$ in the following way:

1°. $n_1 = 1$, $s_1 = 1$,

2°. n_{k+1} is the smallest $l > n_k$ such that

$$t_l < |\inf_{i\leqslant k}(s_i t_{n_i}') - \sup_{i\leqslant k}(s_i t_{n_i})|,$$

3°. s_{k+1} is the smallest m such that

$$\sup_{i\leqslant k}(s_i t_{n_i}) \leqslant m t_{n_{k+1}} < \inf_{i\leqslant k}(s_i t_{n_i}').$$

Let us now consider the intersection

$$A = \bigcap_{k=1}^{\infty} \langle s_k t_{n_k}, s_k t_{n_k}'\rangle.$$

It follows from 1°—3° that every finite intersection of intervals $\bigcap_{l<k} \langle s_l\ t_{n_l},\ s_l\ t'_{n_l}\rangle$ is not empty. Consequently, by Cantor's theorem, there exists a point $t_0 \in \Lambda$, hence $t_{n_k} \leqslant t_0/s_k \leqslant t'_{n_k}$ $(k = 1, 2, \ldots)$. We obtain then by (8)

$$|\psi(t_0/s_k)| > \delta. \tag{9}$$

Since by (6) and 3° $\limsup_{k\to\infty} s_k = \infty$, relation (9) contradicts assumption (ii). The Lemma is thus proved.

If we set now in the Lemma

$$\psi(t) = \frac{\varphi_0(0) - \varphi_0(t)}{t}$$

we obtain from (4) and (5) that $\varphi_0(t)$ is derivable at $t = 0$ and $\varphi_0'(0) = 0$, which completes the proof of our theorem.

We remark that, as Kolmogorov [1] has shown, relation (1) holds for some sequence $\{a_n\}$ if and only if

$$\lim_{b\to\infty} b\ P(|x_i| > b) = 0,$$

where one can take $a_n = \int_{-n}^{n} x\, dF(x)$, and where $F(x)$ is the common distribution function of the x_i. Kolmogorov has also shown that, if (1) holds for a_n equal to some a $(n = 1, 2, \ldots)$, then

$$a = \lim_{n<\infty} \int_{-n}^{n} x\, dF(x). \tag{10}$$

It follows thus from our theorem and from that of Kolmogorov that if $\varphi'(0)$ exists the algebraic moment (10) exists. We have thus a probabilistic proof of this fact which has been proved by Zygmund [3] by an analytical method.

Let us finally remark that our theorem can be used as a device for obtaining sequences of random variables for which the weak law of large numbers holds, while the strong law does not. Indeed, such a sequence is obtained by taking a sequence of independent and equally distributed random variables with a characteristic function $\varphi(t)$ derivable at $t = 0$ and with no finite mean value. The strong law of large numbers will then in virtue of the Kolmogorov theorem [2] not hold for any sequence $\{a_n\}$, while the weak law will hold.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

On Isomorphic Free Algebras

by

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Presented by E. MARCZEWSKI on June 30, 1960

Given a class $\mathfrak{A}$ of abstract algebras having the same operations, we shall denote by $\mathcal{A}_n$ the free $\mathfrak{A}$-algebra [1] with n generators (n finite). Isomorphism of algebras will be denoted by $\simeq$. We announce the following results concerning the distribution of pairs of isomorphic algebras in the sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$.

THEOREM 1. *If k is the smallest integer such that $\mathcal{A}_k \simeq \mathcal{A}_n$ for some $n \neq k$ and $m = k+d$ is the smallest integer satisfying $\mathcal{A}_m \simeq \mathcal{A}_k$, $m \neq k$, then $\mathcal{A}_s \simeq \mathcal{A}_t$ holds for $s \neq t$ if and only if $s \equiv t \pmod{d}$ and $s, t \geqslant k$.*

THEOREM 2. *Given any integers $0 < k < m$ there exists a class of algebras $\mathfrak{A}$ satisfying the assumption of Theorem 1, i.e. with the property that k and m are the smallest integers such that $\mathcal{A}_k \simeq \mathcal{A}_m$, $k \neq m$.*

Theorem 1 is closely related to the results obtained by A. Goetz, E. Marczewski and C. Ryll-Nardzewski, concerning independent generators of an abstract algebra $\mathcal{A}$ [2]. Given an abstract algebra $\mathcal{A}$, let $\mathfrak{A}$ denote the class of all subalgebras of $\mathcal{A}$. It is easily seen that r elements $a_1, \ldots, a_r$ of $\mathcal{A}$ are independent in the sense of E. Marczewski's definition (in [5], sec. 2) if and only if the subalgebra generated by these elements is the algebra $\mathcal{A}_r$ with the free generators $a_1, \ldots, a_r$. Calling a set of independent generators of $\mathcal{A}$, a *basis* and the number of elements of a basis a *rank* of $\mathcal{A}$, it is easily seen that the number r is a rank of $\mathcal{A}$ if and only if $\mathcal{A} \cong \mathcal{A}_r$. Thus Theorem 1 yields in this case an earlier result (cf. [2], Theorem 5) that the finite ranks of any algebra $\mathcal{A}$ form an arithmetic progression.

For given integers $k, m > 0$, we consider the class $\mathfrak{A}$ of all algebras having $m+k$ operations

$$\varphi_i(x_1, \ldots, x_k), \quad \omega_j(x_1, \ldots, x_m), \quad i = 1, \ldots, m; \quad j = 1, \ldots, k$$

which satisfy the "laws"*)

$$\varphi_i(\omega_1(x_1, \ldots, x_m), \ldots, \omega_k(x_1, \ldots, x_m)) = x_i,$$

$$\omega_j(\varphi_1(x_1, \ldots, x_k), \ldots, \varphi_m(x_1, \ldots, x_k)) = x_j.$$

*) similar conditions were considered by Jónsson and Tarski [4].

$\mathfrak{A}$ is a variety in the sense of P. Hall (cf. [3], Def. 2 p. 66) and it has the property stated in Theorem 2. Moreover, $\mathcal{A}_k \simeq \mathcal{A}_r$ holds if and only if r is a rank of $\mathcal{A}_k$. Hence, by Theorem 1, the arithmetic progression $k+qd$ ($q = 0, 1, \ldots$) is the set of all ranks of the algebra $\mathcal{A}_k$. This gives a positive answer to a problem raised by E. Marczewski (cf. [2]), whether every arithmetic progression is the set of all ranks of a certain algebra (this problem was partially solved in [2]).

The proofs will be published in Fundamenta Mathematicae.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
Серия мат., астр. и физ. наук
Том VIII, № 9, 1960

МАТЕМАТИКА

А. АРХАНГЕЛЬСКИЙ

О МЕТРИЗАЦИИ ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВ

Представлено П. С. Александровым 5 июля 1960 г.

В настоящей работе дается несколько новых необходимых и достаточных условий для метризуемости топологических пространств; одни из этих условий связаны с кругом идей теорем Бинга и Нагата-Смирнова, а другие — с новейшей метризационной теоремой П. С. Александрова [1].

Попутно выясняется, в каком случае в любое данное покрытие *) паракомпактного пространства можно вписать локально конечное покрытие, элементы которого принадлежат данной базе пространства.

§ 1. Основные результаты

1. В соответствии с часто применяемой терминологией мы назовем *σ-дизъюнктной* всякую систему множеств, являющуюся счетным объединением своих дизъюнктных подсистем (т. е. подсистем, состоящих из попарно непересекающих множеств.

Теорема 1. *Для того, чтобы топологическое пространство было метризуемым, необходимо и достаточно, чтобы оно было совершенно нормальным и имело σ-дизъюнктную базу.*

2. П. С. Александров называет [1] базу $\mathfrak{B}$ пространства X равномерной, если, каковы бы ни были точка $x \in X$ и ее окрестность Ox, существует не более конечного множества элементов базы, содержащих точку x и пересекающихся с $X \setminus Ox$. Следующее определение является естественным усилением этого определения П. С. Александрова: *база $\mathfrak{B}$ пространства X называется регулярной, если, каковы бы ни были точка $x \in X$ и ее окрестность Ox, существует такая окрестность Ox, существует такая окрестность $O_1x \subset Ox$ этой точки, что не более конечного числа элементов базы $\mathfrak{B}$ одновременно пересекается с O_1x и с $X \setminus Ox$.*

*) Во всей этой работе под покрытием понимается открытое покрытие, под базой топологического пространства — открытая база.

Пространство называется совершенно нормальным, если оно 1) нормально и 2) каждое его замкнутое множество есть G_δ — пересечение счетного числа открытых.

Из этого условия сразу следует, что $[O_1 x] \leqslant Ox$, так что *всякое T_1-пространство с регулярной базой регулярно.* П. С. Александров доказал [2], что пространство X метризуемо тогда и только тогда, когда оно является паракомпактом, имеющим равномерную базу (причем требование паракомпактности может быть заменено требованием коллективной нормальности).

Основным результатом настоящей работы является

Теорема 2. *Для того, чтобы T_1-пространство было метризуемым, необходимо и достаточно, чтобы оно имело регулярную базу.*

Будет доказана и

Лемма 1. *Всякое нормальное пространство с равномерной базой совершенно нормально.*

С другой стороны мы увидим, что во всяком метрическом пространстве имеется равномерная σ-дизъюнктная база. Поэтому Теорема 1 может быть перефразирована так:

Теорема 1′. *Для того, чтобы нормальное пространство было метризуемым необходимо, чтобы у него было равномерная σ-дизъюнктная база и достаточно, чтобы одновременно существовала равномерная база $\mathfrak{V}_1$ и σ-дизъюнктная база $\mathfrak{V}_2$.*

В связи с этим возникает вопрос: *Вытекает ли из существования σ-дизъюнктной базы в нормальном пространстве (в паракомпакте, в бикомпакте) X-метризуемость пространства X?*

Этот вопрос естественно дополняет аналогичные вопросы, поставленные В. Пономаревым и П. С. Александровым в [1] и [4].

3. Переходим к вопросу, вытекает ли из существования в данном T_1-пространстве регулярной базы возможность выделить ее из любой базы данного пространства. Ввиду Теоремы 2 этот вопрос имеет смысл ставить лишь в применении к метризуемым пространствам. Интерес этого вопроса усиливается следующими предложениями:

Теорема 3. *Пусть $\mathfrak{U}$ — база метризуемого пространства X. Для того, чтобы во всякое покрытие пространства X можно было вписать локально конечное покрытие, состоящее из элементов базы $\mathfrak{U}$, необходимо и достаточно, чтобы база $\mathfrak{U}$ содержала регулярную базу.*

Почти очевидна

Теорема 4. *Если метрическое пространство X σ-компактно (т. е. если сумма счетного числа компактов Φ_k, $k = 1, 2, 3, \ldots$), то каждая база этого пространства содержит регулярную базу**).

С другой стороны Н. Константинов показал, что даже в подпространстве числовой прямой, состоящем из всех иррациональных точек, существует база, не содержащая никакой регулярной базы.

*) В самом деле, если $\mathfrak{B}$ — произвольная база пространства X и $\gamma_k \in \mathfrak{B}$ — конечное ε_k — покрытие компакта Φ_k, где $\varepsilon_k \to 0$ при $k \to \infty$, то $\mathfrak{B}_0 = \bigcup_k \gamma_k \leqslant \mathfrak{B}$ есть регулярная база (если $\Phi_k \; \Phi_{k+1}$, $k = 1, 2, \ldots$).

Этот факт объясняет, почему из нашей метризационной Теоремы 2 не выводится сколько нибудь просто классическая теорема Урысона о метризуемости всех нормальных пространств со счетной базой (хотя и выводится легко метризационная теорема для бикомпактов).

§ 2. Доказательства

1. Переходим к доказательству сформулированных выше предложений. Необходимость условий в Теоремах 1 и 1′ вытекает из того, что в любое покрытие γ паракомпакта X можно вписать локально-конечное σ-дизъюнктное покрытие (Теорема Майкла [E. Michael] — см. напр. [3], стр. 156). Применяя этот результат к покрытиям γ_n, $n = 1, 2, 3, \ldots$ метрического пространства X, состоящим соответственно из шаров радиуса $1/n$ и вводя объединение $\mathfrak{B} = \bigcup_n \gamma'_n$, получим искомую равномерную σ-дизъюнктную базу $\mathfrak{B}$.

Лемму 1 доказываем в слегка усиленном виде.

В пространстве X, обладающем равномерной базой, всякое замкнутое множество есть G_δ.

Пусть $\mathfrak{B}$ — равномерная база пространства X. Как доказано П. С. Александровым *) в [1], можно предположить (переходя, если нужно, к базе $\mathfrak{B}_0 \subset \mathfrak{B}$), что $\mathfrak{B} = \bigcup_n \gamma_n$, где γ_n есть покрытие пространства X, причем каждый элемент базы $\mathfrak{B}$, не состоящий из одной изолированной точки, содержится лишь в одном γ_n. Пусть $A \subset X$ замкнуто. Обозначая через $O_n A$ звезду множества A в покрытии γ_n докажем, что $\bigcap_n O_n A = A$. Пусть $x \in \bigcap_n O_n A$. Тогда при любом n найдется $\Gamma_n \in \gamma_n$, содержащее точку x и $\{\Gamma_n\}$ есть база точки x в X; так как $A \wedge \Gamma_n \neq \Lambda$ при любом n, то отсюда следует, что $x \in [A] = A$, чем наше утверждение доказано.

Для доказательства Теорем 1 и 1′ достаточно доказать, что всякое совершенно нормальное пространство X, имеющее σ-дизъюнктную базу $\mathfrak{B} = \bigcup_n \gamma_n$ (каждое $\gamma_n = \{\Gamma_n^\alpha\}$ есть дизъюнктная система множеств) метризуемо. Мы докажем это утверждение, сводя его к метризационной теореме Бинга, т.е построив для пространства X γ-дискретную базу (см. напр. [3], стр. 127). В силу совершенной нормальности пространства X мы можем представить каждое открытое множество $\gamma_n = \bigcup_\alpha \Gamma_n^\alpha$ („тело" системы γ_n) в виде $\widetilde{\gamma}_n = \bigcup_j V_{nj}$, где V_{nj} открыты в X и $[V_{nj}] \subseteq \widetilde{\gamma}_n$. Положим $V_{nj}^\alpha = V_{nj} \cap \Gamma_n^\alpha$ и обозначим систему всех V_{nj}^α, при фиксированных n и j через γ'_{nj}. Покажем, что $\mathfrak{B}' = \bigcup_{n,j} \gamma'_{nj}$ — база пространства X. Пусть точка $x \in X$ и ее окрестность $Ox = \Gamma_{n_0}^{\alpha_0} \in \mathfrak{B}$ выбраны произвольно. Так как $\widetilde{\gamma}_{n_0} = \bigcup_j \widetilde{\gamma}'_{n_0 j}$, то найдется такое j, что $x \in \gamma'_{nj}$. Но тогда $\Gamma_{n_0}^{\alpha_0} \supseteq \Gamma_{n_0}^{\alpha_0} \cap \widetilde{\gamma}'_{n_0 j} = V_{n_0 j}^{\alpha_0} \ni x$, чем утверждение доказано.

*) Это доказательство в сжатом виде воспроизведено ниже (пкт. 3).

Остается доказать, что γ'_{nj} — дискретная система множеств — этим доказательство Теорем 1 и 1′ будет завершено. Наша цель будет достигнута, если для каждой точки $x \in X$ найдем окрестность, пересекающуюся не более, чем с одним элементом V^{α}_{nj} системы γ'_{nj}.

Если $x \in \widetilde{\gamma}_n$, то при некотором α имеем $x \in \Gamma^{\alpha}_n$ и тогда Γ^{α}_n есть нужная нам окрестность точки x. В самом деле, если бы Γ^{α}_n, пересекаясь с элементом $V^{\alpha}_{nj} = \Gamma^{\alpha}_n \cap V_{nj}$ системы γ'_{nj}, пересекалось бы еще с отличным от V^{α}_{nj} элементом $V^{\beta}_{nj} = \Gamma^{\beta}_n \cap V_{nj}$ той же системы, то было бы $\Gamma^{\alpha}_n \cap \Gamma^{\beta}_n \neq \Lambda$, вопреки дизъюнктности системы γ_n.

Если $x \in \widetilde{\gamma}_n$, то в силу соотношения $[V_{nj}] \subseteq \widetilde{\gamma}_n$ у точки x найдется окрестность, именно $X \setminus [V_{nj}]$, не пересекающаяся с V_{nj}, и значит ни с одним $V^{\alpha}_{nj} = V_{nj} \cap \Gamma^{\alpha}_n \in \gamma'_{nj}$.

2. Доказательство Теоремы 2 ведем параллельно с доказательством сформулированной выше (§ 1, пкт. 2) теоремы П. С. Александрова.

Для удобства изложения будем впредь называть равномерные (в смысле П. С. Александрова) базы *точечно-регулярными*. Так как всякая регулярная база является и подавно точечно-регулярной, то *необходимость условий обеих теорем* следует из того, что *всякое метрическое пространство X имеет регулярную базу*. Чтобы ее построить, обозначим через γ'_n покрытие пространства X, состоящее из всех шаров радиуса $1/4n$:

$$\gamma'_n = \left\{ O\left(x, \frac{1}{4n}\right) \right\}.$$

В γ'_n вписываем локально конечное покрытие γ_n. Тогда $\mathfrak{B} = \bigcup_n \gamma_n$ есть очевидно база пространства X. Докажем, что она регулярна. В самом деле, пусть x и Ox даны произвольно. Берем такое натуральное N, чтобы было $\varrho(x, X \setminus Ox) > 1/N$ и окрестность $O_1 x \subseteq O(x, 1/2N)$ пересекающуюся лишь с конечным числом элементов всех покрытий γ_n, где $n \leqslant N$. Тогда $\varrho(O_1 x, X \setminus Ox) > 1/2N$. Элемент какого-либо покрытия γ_n, $n > N$, имея диаметр $\leqslant 1/2n$, может пересекаться лишь с одним из двух множеств $O_1 x$ и $X \setminus Ox$, поэтому элементов базы $\mathfrak{B}$, пересекающихся одновременно с $O_1 x$ и $X \setminus Ox$ может иметься лишь конечное число.

Достаточность. Назовем элемент (точечно) регулярной базы *максимальным*, если он не содержится ни в каком отличном от него элементе этой базы. Очевидно, каждый элемент (точечно) регулярной базы содержится в некотором максимальном элементе, так что множество всех максимальных элементов точечно регулярной базы $\mathfrak{B}$ есть покрытие пространства X.

Лемма 2. *Множество μ всех максимальных элементов U_α точечно регулярной (соответственно регулярной) базы $\mathfrak{B}$ есть точечно (соответственно локально) конечное покрытие.*

В самом деле, пусть $x \in U_1 \in \mu$. Лишь конечное число элементов базы одновременно пересекается с $X \setminus U_1$ и содержит x (пересекается с некоторой фиксированной окрестностью $O_1 x$). Но всякое множество $U_\alpha \in \mu$, содержащее x (пересекающееся с $O_1 x$), пересекается с $X \setminus U_1$ (так как в противном случае это U_α не было бы максимальным). Итак, элементов покрытия μ, содержащих x, соответственно, пересекающихся с $O_1 x$, имеется лишь конечное число. Лемма доказана. Из нее легко выводится.

Следствие. *Пространство с (точечно) регулярной базой (точечно) паракомпактно.*

После этого, достаточность условия Теоремы 2 сразу вытекает из теоремы П. С. Александрова [1]: *паракомпакт X с точечно регулярной базой метризуем.*

3. Наметим вкратце доказательство этой последней теоремы.

Без всякого труда доказывается

Лемма 3. *Если $\mathfrak{B}$ — какая нибудь база пространства X и $\gamma \subset \mathfrak{B}$ — точечно-конечная система, не содержащая в числе своих элементов одноточечных множеств, то $\mathfrak{B} \setminus \gamma$ — снова база.*

Теперь сформулируем

Метризационный критерий В. Пономарева. *Для метризуемости топологического пространства необходимо и достаточно, чтобы оно было паракомпактом со счетной полной системой покрытий* *).

Этот критерий с однаковой легкостью следует как из метризационной теоремы П. С. Александрова и П. С. Урысона [2], так и из теоремы Нагата-Смирнова.

Применяя Лемму 3 к точечно-регулярной базе $\mathfrak{B}$, отщепляем от нее точечно-счетное покрытие γ_1; получаем снова точечно-регулярную базу $\mathfrak{B}_1$, к ней снова применяем Лемму 3 и т. д. В результате получаем последовательность покрытий $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$, не имеющих попарно многоточечных общих элементов. Так как все эти покрытия является подсистемами точечно регулярной базы, то они образуют счетную полную систему. В силу критерия В. Пономарева паракомпакт X метризуем и теорема П. С. Александрова доказана **).

*) Система покрытий $\Sigma = \{\gamma_\alpha\}$ называется полной (П. С. Александров и П. С. Урысон [2]), если какова бы ни была точка x пространства X и содержащий эту точку элемент $\Gamma_\alpha \in \gamma_\alpha$ каждого из покрытий γ_α, система $\{\Gamma_\alpha\}$ образует базу точки x в пространстве X.

**) Это рассуждение есть воспроизведение рассуждения П. С. Александрова; мы даем его, чтобы иметь замкнутое доказательство как метризационной Теоремы 2, так и метризационного критерия П. С. Александрова.

4. Лемма 1 содержит одно из двух утверждений Теоремы 3 (достаточность). Совсем просто доказывается и второе утверждение: пусть база $\mathfrak{B}$ метрического пространства X такова, что во всякое покрытие ω этого пространства можно вписать локально конечное покрытие $\gamma \subseteq \mathfrak{B}$. Беря в качестве ω_n покрытие пространства X шарами радиуса $1/n$, рассмотрим соответствующие локально конечные покрытия $\gamma_n \leqslant \mathfrak{B}$, $n = 1, 2, 3, \ldots$; повторяя наши выше приведенные рассуждения убеждаемся в том, что $\bigcup_n \gamma_n$ есть регулярная база $\mathfrak{B}_0 \subseteq \mathfrak{B}$ пространства X.

С другой стороны, как показывает пример, построенный Н. Константиновым и любезно сообщенный им мне, даже пространство X всех иррациональных точек отрезка $[-1; 1]$ (рассматриваемое как подпространство числовой прямой) имеет базу $\mathfrak{B}$, не содержащую никакой регулярной базы того же пространства.

Отрезок $[-1; 1]$ назовем отрезком нулевого ранга.

Положим

$$a_n = 1 - \frac{1}{2^n}, \qquad a_{-n} = -1 + \frac{1}{2^n}.$$

Через γ_0 обозначим совокупность всех отрезков $[a_{-i}, a_i]$, $i = 1, 2, 3, \ldots$ Отрезки a_k, a_{k+1} (где k — любое целое число) назовем отрезками первого ранга; перенесем подобным преобразованием на каждый из них построение, только что сделанное на отрезке нулевого ранга; при этом система γ_0 перейдет в систему γ_0^k отрезков, лежащих на $[a_k, a_{k+1}]$; положим $\gamma_1 = \bigcup_k \gamma_0^k$. Продолжая этот процесс до бесконечности, получим системы отрезков γ_n, $n = 1, 2, 3, \ldots$, кажда из которых покрывает множество X. Система $\gamma = \bigcup_n \gamma_n$ высекает из X базу, которую обозначим через $\mathfrak{B}$. Покажем, что в любом покрытии ω пространства X элементами базы $\mathfrak{B}$ найдется бесконечная возрастающая последовательность элементов, так что покрытие ω не можем быть локально конечным. Предположим противное. Тогда система μ всех максисальных элементов покрытия ω есть покрытие пространства X.

Система отрезков γ обладает тем свойством, что любые два отрезка из этой системы либо дизъюнктны, либо один из них содержится в другом. Поэтому покрытие μ и высекающая его система отрезков $\Omega \leqslant \gamma$ состоит из дизъюнктных элементов. Если к системе Ω присоединить еще счетную систему одноточечных множеств (состоящих, каждое из одной рациональной точки), то получим представление всего отрезка $[-1; 1]$ в виде суммы счетного числа дизъюнктных множеств, каждое из которых есть либо отрезок, либо одна точка. Так как такое представление невозможно, то наше утверждение доказано.

КАФЕДРА ВЫСШЕЙ ГЕОМЕТРИИ И ТОПОЛОГИИ, МОСКОВСКИЙ ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ.

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A. ARCHANGIELSKIĬ, ON THE METRIZATION OF TOPOLOGICAL SPACES

In this paper several new necessary and sufficient conditions are given for the metrizability of topological spaces. Some of them are based on the main ideas of the Bing and Nagata-Smirnov theorems; others on theorem of Alexandroff on metrization [1] lastly published.

The problem is, further, discussed, under what conditions a locally finite cover, the elements of which belong to the given basis of a space, may be inscribed into an arbitrary given cover of a paracompact space.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHÉMATIQUE

Propriétés limites des solutions des problèmes de Fourier relatifs à l'équation presque linéaire du type parabolique

par

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Présenté par T. WAŻEWSKI le 5 juillet 1960

1. Introduction. Soit G un domaine borné de l'espace $\mathcal{E}^m$ des variables $x_1, x_2, \ldots, x_m$, $\bar{G}$ — sa fermeture et $\dot{G}$ — sa frontière. Désignons par D et D_T les produits topologiques $G\times(0, \infty)$ resp. $G\times(T, \infty)$, où $T > 0$, par σ et σ_T — la surface latérale $\dot{G}\times(0, \infty)$ du D, resp. $\dot{G}\times(T, \infty)$ du D_T.

Nous allons considérer l'équation presque linéaire du type parabolique

$$(1) \qquad F(u) \equiv \sum_{ij=1}^{m} a_{ij}(X, t)\, u''_{x_i x_j} + \sum_{k=1}^{m} b_k(X, t)\, u'_{x_k} - u'_t = f(X, t, u),$$

où X est le point variable aux coordonnées $x_1, x_2, \ldots, x_m$ de l'espace $\mathcal{E}^m$. Dans le cas, où les limites

$$(A) \qquad \lim_{t\to\infty} a_{ij}(X, t = a_{ij}(X), \qquad \lim_{t\to\infty} b_k(X, t) = b_k(X)$$

existent et la convergence est uniforme par rapport à $X \in \bar{G}$, nous allons considérer aussi l'équation du type elliptique

$$(2) \qquad \sum_{ij=1}^{m} a_{ij}(X, t)\, v''_{x_i x_j} + \sum_{k=1}^{m} b_k\, v'_{x_k} = f(X, v).$$

Soit $u(X, t)$ une solution de l'équation (1) régulière dans $\bar{D}$*) et $v(X)$ une telle solution de (2). Dans la première partie du présent travail nous allons établir certaines conditions suffisantes pour qu'une solution $u(X, t)$ de (1) régulière dans $\bar{D}$ et telle que $\lim_{t\to\infty} u(X, t) = v(X)$ uniformément par rapport à $X \in \dot{G}$, satisfasse à la même condition uniformément par rapport à $X \in \bar{G}$.

Dans la seconde partie nous considérons un problème analogue relatif aux solutions du troisième problème de Fourier.

Sous des hypothèses plus restrictives ou bien dans des cas particuliers ces problèmes ont été déjà traités par: A. Friedman [1], [2], H. Grużewska [6],

*) C'est à dire continue dans l'ensemble $\bar{D}$ et admettant des derivées du second ordre par rapport aux variables $x_1, \ldots, x_m$ et du premier ordre par rapport à t continues dans D.

M. Krzyżański [3], [4], H. Murakami [8], R. Narasihman [9], P. Zeragia [12], I. Królikiewicz [5].

2. Dans le présent travail nous nous basons sur un lemme constituant un cas particulier d'un théorème du travail [7] de J. Szarski, à savoir:

LEMME. *Soit* $u(X, t)$ *une solution de l'équation linéaire parabolique*

$$\sum_{ij=1}^{m} \bar{a}_{ij}(X, t)\, u''_{x_i x_j} + \sum_{k=1}^{m} \bar{b}_k(X, t)\, u'_{x_k} - u'_t + \bar{c}(X, t)u = \bar{f}(X,t) \tag{3}$$

où les fonctions $\bar{a}_{ij}$, b_k, $\bar{c}$, $\bar{f}$ *sont définies dans* $\bar{D}_T$; *nous supposons que dans ce domaine on a* $c(X, t) \leqslant 0$. *Si* $u(X, t) \leqslant 0$ (*resp.* $u(X, t) \geqslant 0$) *sur* $\bar{G}+\sigma_T$ *et si aux points de* D_T, *où* $u(X, t) > 0$ (*resp.* $u(X, t) < 0$) *on a*

$$\bar{f}(X, t) \geqslant 0 \ (\textit{resp.}\ \bar{f}(X, t) \leqslant 0), \tag{4}$$

alors

$$u(X, t) \leqslant 0 \ (\textit{resp.}\ u(X, t) \geqslant 0) \ \textit{dans}\ \bar{D}_T.$$

Dans la suite nous supposons que:

(B) dans l'équation (1) la fonction $f(X, t, z)$ est définie dans l'ensemble $\bar{D} \times (-\infty, \infty)$ et y satisfait à la condition suivante: pour tous z' et z'' tels que $z' \leqslant z''$ on a l'inégalité suivante

$$f(X, t, z') \leqslant f(X, t, z''); \tag{5}$$

(C) les coefficients $a_{ij}(X, t)$, $b_k(X, t)$ sont définis et la forme caractéristique $\sum_{ij=1}^{m} a_{ij}\lambda_i\lambda_j$ est définie positive dans $\bar{D}$.

Nous allons démontrer le théorème suivant:

THÉORÈME 1. *Admettons les hypothèses* (*B*), (*C*). *Soit* $u(X, t)$ *une solution de* (1) *régulière dans* D *et telle que*

$$\lim_{t\to\infty} u(X, t) = 0 \tag{6}$$

uniformément par rapport à $X \in \dot{G}$. *Supposons qu'il existe une fonction* $V(X, t)$ *régulière dans l'ensemble* D *et satisfaisant aux conditions suivantes: il existe un nombre* $T > 0$ *tel que l'on a dans l'ensemble* D_T *des inégalités*

$$V(X, t) > 0, \tag{7}$$

$$\lim_{t\to\infty} V(X, t) = 0 \ \textit{uniformément pour}\ x \in \bar{G}, \tag{8}$$

$$F(V) \leqslant -|f(X, t, 0)|. \tag{9}$$

Ces hypothèses étant admises on a $\lim_{t\to\infty} u(X, t) = 0$ *uniformément dans* $\bar{G}$.

Démonstration. Nous formons la fonction auxiliaire $v_1(X, t) = u(X, t) - \varepsilon - KV(X, t)$, ε étant un nombre positif arbitraire et $K > 1$ un nombre que nous allons choisir convenablement dans la suite.

En vertu de (6) il existe un nombre $T_1 > T$ tel que l'on a $|u(X, t)| < \varepsilon$ sur σ_{T_1}. D'après (7) nous pouvons choisir un nombre $K > 1$ suffisamment grand pour

que $v_1(X, T_1) \leqslant 0$, où $X \in \bar{G}$. La fonction $v_1(X, t)$ est non positive sur $\dot{D}_{T_1}$, elle satisfait à l'équation linéaire de la forme (3) dans laquelle $\bar{c}(X, t) \equiv 0$. Aux points $(X, t) \in D_{T_1}$, où $v_1(X, t) > 0$, on a d'après (5) et (9)

$$\bar{f}(X, t) = f(X, t, u(X, t)) - KF(V) \geqslant 0.$$

En vertu du Lemme, $v_1(X, t) \leqslant 0$ dans $\bar{D}_{T_1}$ entier.

En posant $v_2(X, t) = u(X, t) + \varepsilon + KV(X, t)$ et en appliquant un raisonnement identique a celui exposé ci-dessus on démontre qu'on a $v_2(X, t) \geqslant 0$ pour $(X, t) \in D_{T_1}$. On deduit des inégalités $v_1(X, t) \leqslant 0$ et $v_2(X, t) \geqslant 0$ l'inégalité suivante:

$$|u(X, t)| \leqslant \varepsilon + KV(X, t)$$

aux points de D_{T_1}. En vertu de (8), il existe $T_2 > T_1$ tel que $|u(X, t)| < 2\varepsilon$ dans $\bar{D}_{T_2}$. On a donc $\lim_{t \to \infty} u(X, t) = 0$ uniformément dans $\bar{G}$.

3. Nous allons construire la fonction $V(X, t)$ en conservant les hypothèses sur les coefficients de l'équation (1) et la fonction $f(X, t, z)$ exposées dans par. **2** et en y ajoutant celles qui vont suivre:

(D) la fonction $f(X, t, z)$ est continue par rapport à $(X, t) \in \bar{D}$ et on a

$$\lim_{t \to \infty} f(X, t, 0) = 0 \tag{10}$$

uniformément par rapport à $X \in \bar{G}$, et, en outre, $f(x, t, 0) \not\equiv 0$ dans D^*);

(E) il existe un indicateur $i \leqslant m$ tel que l'on a $a_{ii}(X, t) \geqslant a_0 > 0$ pour $(X, t) \in D$ et $b_i(X, t) \leqslant b_0$, où $b_0 > 0$.

Posons

$$\sup_{x \in G,\, \tau \geqslant t} |f(X, \tau, 0)| = p(t). \tag{11}$$

La fonction $p(t)$ est continue, non croissante dans $(0, \infty)$ et, en vertu de (10), on a

$$\lim_{t \to \infty} p(t) = 0. \tag{12}$$

On introduit la fonction $w(X) = 2 - \exp(c - \alpha x_i)$, [9], où nous choisissons le nombre c de façon à avoir $c - \alpha x_i < 0$ pour $X \in \bar{G}$ et le nombre $\alpha > 0$ de sorte que

$$F(w) \leqslant -1; \tag{13}$$

on a aussi

$$1 \leqslant w(X) \leqslant 2. \tag{14}$$

Posons

$$V(X, t) = w(X) \cdot I(t), \tag{15}$$

où

$$I(t) = \int_0^t p(\tau) \exp[\lambda(\tau - t)]\, d\tau \text{ **)}. \tag{16}$$

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*) Le cas où $f(x, t, 0) \equiv 0$ a été considéré dans mon travail antérieur [5].

**) Remarquons que la fonction $I(t)$ que nous avons introduit pour construire la fonction $V(X, t)$ conforme aux hypothèses du Théorème 1 est analogue à la transformée de Laplace-Picone [10], [11] et n'en diffère que par le signe de l'exposant. Cette fonction a été appliquée aussi dans [12].

D'après (13), on a

$$F(V) \leqslant [\lambda w(X) - 1] I(t) - p(t),$$

où, en vertu de (14), on peut choisir le nombre $\lambda > 0$ de sorte à avoir $\lambda w(X) < 1$. Alors, en vertu de (11), nous avons:

$$F(V) \leqslant -|f(X, t, 0)|.$$

Il résulte de (12) que l'on a aussi $\lim_{t\to\infty} V(X, t) = 0$ uniformément dans $\bar{G}$. Donc, la fonction $V(X, t)$ determinée par (15) satisfait aux hypothèses (7), (8) et (9) du Théorème 1. Il s'en suit le théorème suivant:

THÉORÈME 2. *Les hypothèses* (*B*), (*C*) *du Théorème* 1 *restent valables. Nous admettons, en outre, les hypothèses* (*D*) *et* (*E*). *Alors la solution de l'équation* (1) *régulière dans* $\bar{D}$, *qui pour* $t \to \infty$ *tend uniformément vers zéro sur* $\dot{G}$, *tend aussi vers zéro uniformément dans G entier.*

4. Supposons maintenant que les coefficients $a_{ij}(X)$ et $b_k(X)$ dépendent seulement des variables $x_1, \ldots, x_m$ et que

(C′) les coefficients a_{ij} *ei* b_k sont définis et la forme caractéristique $\sum_{ij=1}^{m} a_{ij} \lambda_i \lambda_j$ est définie positive dans $\bar{G}$;

(D′) La fonction $f(X, t, z)$ est continue par rapport à $(X, t) \in D$ et satisfait à la condition $\lim_{t\to\infty} f(X, t, z) = f(X, z)$ uniformément par rapport à $(X, t) \in G \times H$, pour tout ensemble compact du H $(-\infty, \infty)$.

Nous allons démontrer le théorème suivant:

THÉORÈME 3. *Supposons que les hypothèses* (*B*), (*C′*) *et* (*D′*) *soient verifiés. Soit* $u(X, t)$ *une solution de* (1) *régulière dans D, telle que* $\lim_{t\to\infty} u(X, t) = \varphi(X)$ *uniformement par rapport à* $X \in \dot{G}$ *et* $v(X)$ *une solution de* (2) *régulière dans G et telle que* $v(X) = \varphi(X)$ *sur* $\dot{G}$. *Supposons, en outre, qu'il existe une fonction* $V(X, t)$ *régulière dans l'ensemble* $\bar{D}$ *satisfaisant à la condition suivante; il existe un nombre* $T > 0$ *tel que dans l'ensemble* D_T *cette fonction satisfait aux conditions* (7), (8) *et à l'inégalité*

$$F(V) \leqslant -|f(X, t, v(X)) - f(X, v(X))|. \tag{17}$$

Alors on a $\lim_{t\to\infty} u(X, t) = v(X)$ *uniformément par rapport à* $X \in G$.

Il est aisé de voir qu'on peut démontrer ce Théorème en procédant de la même façon que dans la démonstration du Théorème 2, si l'on pose $v_1(X, t) = u(X, t) - v(X) - \varepsilon - KV(X, t)$ et $v_2(X, t) = u(X, t) - v(X) + \varepsilon + KV(X, t)$.

Si l'on admet, en outre, l'hypothèse (E), la fonction $V(X, t)$ peut être déterminée par les formules (15) et (16,) où l'on pose

$$p(t) = \sup_{X\in\bar{G},\, \tau\geqslant t} |f(X, \tau, v(X)) - f(X, v(X))|.$$

Cette fonction satisfait aux conditions (7), (8) et (17).

On a donc le théorème suivant

THÉORÈME 4. *Admettons les hypothèses* (*B*), (*C'*), (*D'*) *et* (*E*). *Alors la solution de* (1), *régulière dans* $\bar{D}$, *qui tend pour* $t \to \infty$ *vers une solution régulière de* (2) *uniformément pour* $X \in \bar{G}$, *tend aussi pour* $t \to \infty$ *vers cette fonction uniformément par rapport à* $X \in \bar{G}$.

Remarque 1. Si la fonction $v(X)$ satisfait aux hypothèses plus restrictives, à savoir si les derivées $v''_{x_i x_j}$, v'_{x_k} sont bornées dans $\bar{G}$ pour $k, i, j = 1, \ldots m$, on peut démontrer des théorèmes analogues aux Théorèmes 3 et 4 dans les cas, où les coefficients a_{ij}, b_k de l'équation (1) dépendent non seulement de X, mais aussi du temps t, en admettant l'hypothèse (A) (par. 1).

Remarque 2. Il est facile de voir que dans le cas de l'équation

$$\sum_{ij=1}^{n} a_{ij}(X, t)\, u''_{x_i x_j} - u'_t = f(X, t, u\, u'_{x_1}, \ldots, u'_{x_m}),$$

où la forme quadratique $\sum_{ij=1}^{m} a_{ij}\, \lambda_i \lambda_j$ est uniformément définie positive dans $\bar{D}$, les Théorèmes 2 et 3 restent valables sous l'hypothèse analogue à celle admise par R. Narasihman dans [9], à savoir:

il existe une constante $K > 0$ et une fonction $g(X, t, z)$ telles que pour $|u| < K$ on a

$$u \cdot f(X, t, u'_{x_1}, \ldots, u'_{x_m}) \geqslant \lambda (g(X, t, u^2) - \sum_{i=1}^{m} u'^2_{x_i}),$$

dans D, λ étant un nombre positif et $g(X, t, z)$ une fonction qui satisfait à l'hypothèse (8).

5. J. Szarski [7] a démontré aussi un théorème sur les inégalités différentielles aux dérivées partielles relatif au troisième problème de Fourier. En appliquant ce théorème. nous pouvons démontrer les théorèmes analogues aux précédents, relatifs à ce problème.

Posons

$$L(u) = \frac{du}{dl} - h(X, t)\, u - g(X, t), \tag{18}$$

$h(X, t)$ et $g(X, t)$ étant des fonctions définies sur σ et l une demi-droite issue du point $(X, t) \in \sigma$ orthogonale à l'axe t, pénétrant à l'intérieur de D. Nous supposons que l'on a $h(X, t) \geqslant h_0 > 0$ sur σ.

THÉORÈME 5. *Admettons que les hypothèses* (*B*) *et* (*C*) *restent valables. Soit* $u(X, t)$ *une solution de* (1) *birégulière dans* $\bar{D}$, *satisfaisant à la condition aux limites* (18). *Soit*

$$\lim_{t \to \infty} g(X, t) = 0 \tag{19}$$

uniformément dans $\dot{G}$. S'il existe une fonction $V(X, t)$ birégulière $\bar{D}$ et telle que pour $T > 0$ suffisamment grand elle satisfait aux conditions (7), (8), (9), *et aussi à*

(20) $L(V) < 0$ *dans* D_T,

alors on a $\lim_{t\to\infty} u(X, t) = 0$ *uniformément dans* G *).

On peut déterminer une fonction $V(X, t)$ conforme aux hypothèses du Théorème 5 en posant $V(X, t) = [w(X)+d]\, I(t)$ (voir par. 3) d étant un nombre constant positif que l'on détermine de façon que la condition (20) soit verifiée.

Il en résulte le théorème suivant:

THÉORÈME 6. *On admet les hypothèses* (*B*), (*C*), (*D*), *et* (*E*). *Si $u(X, t)$ est une solution de* (1) *birégulière dans D satisfaisant à la condition* (18), *où l'on a* (19) *alors* $\lim_{t\to\infty} u(X, t) = 0$ uniformément dans $\bar{G}$.

Remarque. Si la direction de la demi-droite l ne dépend pas de la variable t, si, de plus, on a $\lim_{t\to\infty} h(X, t) = h(X)$ et $\lim_{t\to\infty} g(X, t) = g(X)$ uniformément par rapport à $X \in G$, et si, en outre, les coefficients de l'équation (1) et leur deuxième membre satisfont aux hypothèses (C′), (B) et (D′), alors on peut démontrer que la solution $u(X, t)$ de l'équation (1) satisfaisant à (18) tend uniformément dans G vers la solution $v(X)$ de (2) satisfaisant à la condition

$$\frac{dv}{dl} + h(X)\, v = g(X) \quad \text{sur } \sigma.$$

Les résultats exposés dans la note présente ont été obtenus sous la direction du Professeur M. Krzyżański. Je lui exprime ici ma gratitude profonde.

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*) A Friedman [2] considére l'allure asymptotique de la solution de l'équation linéaire homogène qui satisfait à la condition aux limites non linéaire. Mais ces théorèmes ne contiennent pas en général les résultats présentés ci-dessus (voir [2] p. 171).

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHÉMATIQUE

Sur la représentation d'une variété par une hypersurface convexe de l'espace euclidien

par

M. ROCHOWSKI

Présenté par E. MARCZEWSKI le 12 juillet, 1960

J. Hadamard ([1], p. 352) a montré que, si $x : M^n \to E^{n+1}$ est une représentation régulière d'une hypersurface $x(M^n)$ dans l'espace euclidien E^{n+1} et si la courbure K de Gauss-Kronecker de $x(M^n)$ est partout positive, cette hypersurface est convexe. Bien que le théorème ait été démontré pour $n = 2$, sa démonstration s'étend à n quelconque. S. S. Chern et R. K. Lashof [2] ont démontré que le théorème d'Hadamard tombe en défaut pour $n > 2$, si l'on remplace l'inégalité $K > 0$ par la condition $K \geqslant 0$.

Je vais montrer dans cette note que la thèse du théorème d'Hadamard demeure vraie, si l'on suppose que la seconde forme fondamentale est non-positive en chaque point de l'hypersurface ce qui veut dire qu'elle prend des valeurs $\leqslant 0$ pour tout vecteur r.

1. Soit K^{n+1} une boule euclidienne de dimension $n+1$ ayant pour centre l'origine des coordonnées. Nous allons considérer sur K^{n+1} la famille des fonctions

$$\{g(x, \varepsilon)\}, \quad \varepsilon > 0, \quad x \in K^{n+1}, \tag{1}$$

où l'on a posé

$$g(x, \varepsilon) = \varepsilon(1 - x_1^2 - \ldots - x_{n+1}^2).$$

Ces fonctions jouissent des propriétés suivantes:

1° $0 \leqslant g(x, \varepsilon) \leqslant \varepsilon, \quad g(x, \varepsilon) = 0 \quad$ pour $\quad x \in S^n = Fr(K^{n+1})$,

2° la forme $\displaystyle\sum_{\mu, \nu=1}^{n+1} \frac{\partial^2 g}{\partial x_\mu \partial x_\nu} t_\mu t_\nu = -2\varepsilon \sum_{\mu=1}^{n+1} t_\mu^2$

est définie négative pour $\varepsilon > 0$,

3° $\displaystyle\lim_{\varepsilon \to 0} \sum_{\mu, \nu=1}^{n+1} \frac{\partial^2 g}{\partial x_\mu \partial x_\nu} t_\mu t_\nu = 0.$

2. Soit M^n une variété C^∞ fermée et orientable à n dimensions et $x : M^n \to K^{n+1}$ une représentation de classe C^∞ de M^n dans l'intérieur de K^{n+1}; nous supposons que cette représentation soit régulière, c'est-à-dire que la matrice de la représentation x soit de rang n en tout point $p \in M^n$.

Supposons que la seconde forme fondamentale de l'hypersurface $S = x(M^n)$ soit non-positive en tout point de S. Il existe alors pour tout point $p \in M^n$ un voisinage $U(p)$ tel, que $x(U(p))$ est une hypersurface convexe dans un système convenablement choisi, des coordonnées dans K^{n+1}. En effet, la représentation x étant régulière, on peut supposer que, dans un voisinage $U(p)$ suffisamment petit, elle a la forme suivante

$$x_{n+1} = f(x_1, \ldots, x_n), \tag{2}$$

$$f(0) = 0, \quad \left(\frac{\partial f}{\partial x_i}\right)_0 = 0. \tag{3}$$

Or, il résulte de l'hypothèse sur la seconde forme fondamentale de S, que la seconde différentielle de la fonction (2) est non-positive pour desaccroissements arbitraires des variables indépendantes. La fonction (2) satisfait donc à la condition 2° du par. **1**, d'où il résulte que la fonction f et l'hypersurface $x(U(p))$ sont convexes. La variété M^n peut donc être orientée de telle manière que le vecteur normal, positivement orienté par rapport à l'hypersurface S, soit dirigé en dehors du demi-espace ayant pour frontière l'hyperplan tangent au point $x(p)$ et contenant $x(U(p))$.

3. Nous allons introduire une application auxiliaire

$$\bar{x}: M^n \to K^{n+1}$$

de M^n, dans l'intérieur de K^{n+1}, définie par la formule suivante

$$\bar{x}(p, \varepsilon) = x(p) + \xi(p) \cdot g(x(p), \varepsilon) \tag{4}$$

$\xi(p)$ désignant le vecteur unité normal en $x(p)$ à l'hypersurface $\bar{x}(M^n)$ et g une fonction de la famille (1).

D'après la condition 1° du par. **1** on a

$$\lim_{\varepsilon \to 0} S_\varepsilon = S, \tag{5}$$

où $S_\varepsilon = \bar{x}(M^n)$. L'application x et la fonction g étant de classe C^∞ il en est de même pour l'application $\bar{x}$.

LEMME. *L'application $\bar{x}$ est régulière.*

Démonstration. Les composantes du vecteur normal de l'hypersurface (2) sont définies par les formules

$$\xi_i(p) = -\frac{\dfrac{\partial f}{\partial x_i}}{\sqrt{1 + \sum_{i=1}^{n} \left(\dfrac{\partial f}{\partial x_i}\right)^2}}, \qquad i = 1, \ldots, n$$

$$\xi_{n+1}(p) = \frac{1}{\sqrt{1 + \sum_{i=1}^{n} \left(\dfrac{\partial f}{\partial x_i}\right)^2}}.$$

Il en résulte que l'application $\bar{x}$, exprimée en coordonnées locales x_i, a la forme suivante

$$(6)\quad \begin{cases} \bar{x}_i = x_i - \dfrac{\dfrac{\partial f}{\partial x_i}}{\sqrt{1+\sum_{i=1}^{n}\left(\dfrac{\partial f}{\partial x_i}\right)^2}}\, g'(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \varepsilon), \quad i = 1, \ldots, n \\ \bar{x}_{n+1} = f(x_1, \ldots, x_n) + \dfrac{g'(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \varepsilon)}{\sqrt{1+\sum_{i=1}^{n}\left(\dfrac{\partial f}{\partial x_i}\right)^2}}, \end{cases}$$

g' se déduisant de g au moyen d'un changement des variables que nous avons effectué pour obtenir les Eq. (2) et (3). Pour prouver que l'application $\bar{x}$ est régulière, il suffit de montrer que la valeur D_0 du jacobien

$$D = \left| \frac{\partial \bar{x}_i}{\partial x_j} \right| \quad i, j = 1, 2, \ldots, n$$

au point $x_i = 0$ est différente de zéro. En vertu des formules (6) on a

$$D_0 = \left| \delta_{ij} - \frac{\partial^2 f}{\partial x_i \partial x_j} g' \right|,$$

où les valeurs de $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ et de g' doivent être prises pour $x_i = 0$. En développant le second membre, on trouve

$$D_0 = 1 - A_1 g' + A_2 g'^2 - \ldots + (-1)^n A_n g'^n,$$

A_k désignant la somme des mineurs principaux de degré k du déterminant

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

de la forme non-positive

$$\sum \frac{\partial^2 f}{\partial x_i \partial x_j} t_i t_j.$$

Puisqu'on a $(-1)^k A_k \geqslant 0$ et $g' > 0$, il en résulte $D_0 \geqslant 1$, ce qui démontre le lemme.

L'application (4) étant, comme il vient d'être démontré, régulière, l'hypersurface S peut être présentée, dans un voisinage d'un point quelconque $p \in M^n$, par la formule

$$(7)\quad \bar{x}_{n+1} = f(x_1, \ldots, x_n) + \frac{g'(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \varepsilon)}{\sqrt{1+\sum_{i=1}^{n}\left(\dfrac{\partial f}{\partial x_i}\right)^2}},$$

où les x_i doivent être exprimées en fonction des variables $\bar{x}_i$ au moyen de la première ligne des formules (6).

4. Théorème. *Si la seconde forme fondamentale de l'hypersurface $S = x(M^n)$ est partout non-positive, la variété M^n est plongée dans E^{n+1} comme une hypersurface convexe.*

Démonstration. Nous allons montrer, en premier lieu, que la seconde forme fondamentale de l'hypersurface S_ε est définie négative pour tout $\varepsilon > 0$.

En tenant compte des égalités (3) on déduit de la formule (7) l'expression suivante

$$d^2 \bar{x}_{n+1} = d^2 f + d^2 g' - g' d^2 \left(\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right).$$

Rappelons que la différentielle $d^2 f$ est non-positive (par. **2**) et qu'en vertu de la propriété 2^o du par. **1**, la forme $d^2 g'$ est définie négative. Nous avons, en outre,

$$d^2 \left(\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right) = 2 \sum_j \left(\sum_i \frac{\partial x_i \, \partial x_j}{\partial^2 f} \, dx_i \right)^2 \geqslant 0.$$

On voit ainsi que la différentielle $d^2 x_{n+1}$ exprimée au moyen des variables x_i devient une forme définie négative en tout point $\bar{x}(p)$ de l'hypersurface S_ε ($\varepsilon > 0$). L'expression $d^2 \bar{x}_{n+1}$ étant une forme définie négative il en suit, que la seconde forme fondamentale de l'hypersurface S_ε est aussi définie négative. Il en résulte que la courbure de Gauss-Kronecker de S_ε est positive et que par suite, en vertu du théorème d'Hadamard, cette hypersurface est convexe pour $n \geqslant 2$. On en conclut, en rapprochant l'égalité (5), que l'hypersurface $S = x(M^n)$ est convexe pour tout $n \geqslant 2$.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

Some Remarks Concerning the Position of Sets in a Space

by
K. BORSUK

Presented on July 13, 1960

In the present note I introduce some simple notions concerning the position of closed sets in a space.

1. Homological domination. Let X_1 and X_2 be two closed subsets of a space X and let i_1 and i_2 denote the inclusion maps:

$$i_1\colon X_1 \to X; \quad i_2\colon X_2 \to X.$$

We say that the set X_2 *homologically dominates* the set X_1 in the space X, written

$$X_1 \underset{hl}{\leqslant} X_2 \text{ in } X, \tag{1}$$

provided the homomorphisms i_{1*}, i_{2*} induced on the homology groups by the inclusions i_1, i_2:

$$i_{1*} : H_n(X_1, \mathfrak{A}) \to H_n(X, \mathfrak{A}),$$

$$i_{2*} : H_n(X_2, \mathfrak{A}) \to H_n(X, \mathfrak{A}),$$

where n is an arbitrary integer and $\mathfrak{A}$ is an arbitrary Abelian group of coefficients, satisfy the condition

$$\operatorname{Im} i_1 \subset \operatorname{Im} i_2.$$

It is clear that the relation $\underset{hl}{\leqslant}$ is reflexive and transitive, i.e. $X_1 \underset{hl}{\leqslant} X_2$ in X and $X_2 \underset{hl}{\leqslant} X_3$ in X imply $X_1 \underset{hl}{\leqslant} X_3$ in X. Let us observe that

$$X_1 \underset{hl}{\leqslant} X \text{ in } X \text{ for every closed subset } X_1 \text{ of } X.$$

2. Homotopic domination. We say that the set X_2 *homotopically dominates* the set X_1 in the space X, written

$$X_1 \underset{ht}{\leqslant} X_2 \text{ in } X, \tag{2}$$

provided there exists a continuous map

$$\alpha : X_1 \to X_2$$

such that i_1 is homotopic to $i_2\,\alpha$. Then the map α is said to *realize* the relation (2).

The homotopy of i_1 with $i_2\alpha$ implies that the induced homomorphisms i_{1*} and $i_{2*}\alpha_*$ coincide. It follows that Im i_{1*} is a subgroup of Im i_{2*}, hence

$$X_1 \underset{ht}{\leqslant} X_2 \text{ in } X \quad \text{implies} \quad X_1 \underset{hl}{\leqslant} X_2 \text{ in } X. \tag{3}$$

It is clear that the relation $\underset{ht}{\leqslant}$, similarly to the relation $\underset{hl}{\leqslant}$, is reflexive and transitive. Moreover,

$$X_1 \underset{ht}{\leqslant} X \text{ in } X \quad \text{for every closed subset} \quad X_1 \text{ of } X.$$

3. Homological and homotopic equivalence. Let X, X_1, X_2, i_1, i_2 be as before. We say that X_1 and X_2 are *homologically equivalent* in X, in symbols

$$X_1 \underset{hl}{\equiv} X_2 \text{ in } X,$$

provided both relations $X_1 \underset{hl}{\leqslant} X_2$ in X and $X_2 \underset{ht}{\equiv} X_1$ in X hold. Then Im $i_{1*} \subset$ Im i_{2*} and Im $i_{2*} \subset$ Im i_{1*}, hence Im $i_{1*} =$ Im i_{2*}. Conversely, the relation Im $i_{1*} =$ = Im i_{2*} implies the homological equivalence of X_1 and X_2 in X. Consequently,

$$X_1 \underset{hl}{\equiv} X_2 \text{ in } X \Leftrightarrow \text{Im } i_{1*} = \text{Im } i_{2*}. \tag{4}$$

In a similar way we define the homotopic equivalence of X_2 and X_2 in X by both relations: $X_1 \underset{ht}{\equiv} X_2$ in X and $X_2 \underset{ht}{\equiv} X_1$ in X.

It is clear that both relations $\underset{hl}{\equiv}$ and $\underset{ht}{\equiv}$ are reflexive, symmetric and transitive. Moreover, it follows by (3) that

$$X_1 \underset{ht}{\equiv} X_2 \text{ in } X \quad \text{implies} \quad X_1 \underset{hl}{\equiv} X_2 \text{ in } X. \tag{5}$$

Example 1. Consider in the Euclidean 3-space E^3 a point p, two 2-spheres S_1 and S_2 and the surface T of an anchor ring. Let us assume that p lies in the bounded component of $E^3 - S_1$, in the unbounded component of $E^3 - S_2$ and in the bounded component of $E^3 - T$. Then we easily see that:

$$S_1 \underset{hl}{\equiv} T \underset{ht}{\equiv} S_1 \cup S_2 \text{ in } E^3 - (p),$$

$$S_1 \underset{hl}{\not\equiv} S_2 \text{ in } E^3 - (p),$$

$$T \underset{ht}{\leqslant} S_1 \text{ in } E^3 - (p) \text{ and } S_1 \underset{ht}{\not\equiv} T \text{ in } E^3 - (p).$$

Example 2. Let P_2 be the projective plane, $A \subset P_2$ be a projective straight line, B be the union of all 1-dimensional simplexes of a triangulation T of P_2 and C be the boundary of a triangle belonging to T. Then we easily see that:

$$A \underset{ht}{\equiv} B \text{ in } P_2, \quad C \underset{ht}{\leqslant} A \text{ in } P_2, \quad A \underset{ht}{\equiv} C \text{ in } P_2.$$

4. k-skeletons of space X. A closed subset X_1 of a space X is said to be a *homological k-skeleton* of X provided that dim $X_1 \leqslant k$ and for each closed subset X_2 of X with dim $X_2 \leqslant k$, we have $X_2 \underset{hl}{\leqslant} X_1$ in X.

A closed subset X_1 of X is said to be a *homotopic k-skeleton* of X, provided $\dim X_1 \leqslant k$ and for each closed subset X_2 of X with $\dim X_2 \leqslant k$, we have $X_2 \underset{ht}{\leqslant} X_1$ in X. We see at once that:

(6) All homological k-skeletons of X are homologically equivalent in X.

(7) All homotopic k-skeletons of X are homotopically equivalent in X.

(8) Each homotopic k-skeleton of X is a homological k-skeleton of X.

It is easy to give examples of compacta for which the homological k-skeleton does not exist:

Example 3. Let X be the union of a 2-dimensional disk Q and of an infinite sequence $\{S_n\}$ of circles disjoint one to the other and such that the diameter of S_n is $1/n$, the set $Q \cap S_n$ consists of one point a_n and the sequence $\{a_n\}$ is dense in Q. For every closed subset X_1 of X with $\dim X_1 \leqslant 1$ there exists an index n_0 such that $S_{n_0} \subset X - X_1$. It follows at once that the relation $S_{n_0} \underset{hl}{\leqslant} X_1$ does not hold. Consequently X_1 is not a homological 1-skeleton of X.

For ANR-sets the problem of existence of homological and of homotopic k-skeletons remains open. In the next No. we solve it for some more special class of spaces.

5. Homotopic k-skeletons for ANR-sets with the property (Δ). A space X is said to have the property (Δ) [3], provided there exists for every point p of X and for every neigbourhood U of p a neighbourhood V of p such that each compact subset A of V can be contracted continuously to the point p over a set $B \subset U$ of dimension $\leqslant \dim A + 1$. In particular, every polytope has the property (Δ) (see [3], p. 87).

In the sequel we need the following property (see [3] p. 79) of compacta with the property (Δ):

(9) *If X is a compactum with property (Δ), then, for every continuous map f of a compactum Z into X and every $\varepsilon > 0$, there exists a continuous map g of Z into X such that $\varrho(f, g) < \varepsilon$ and* $\dim g(Z) \leqslant \dim Z$.

Moreover, we shall use in the sequel the following elementary lemma:

(10) *Let T be a triangulation of the polytope W and Z a compactum of dimension $\leqslant k$. Then for every map $f \in W^Z$ there exists a map $g \in W^Z$, homotopic to f in W^Z, which maps Z into the union of all at most k-dimensional simplexes of T.*

In order to obtain the map g, we modify step by step the map f, using the standard process of P. Alexandroff (see, for instance, [1], p. 114) by which we shift the values lying in the interiors of simplexes with dimensions $> k$ onto boundaries of those simplexes. As an immediate consequence of the lemma (10) we have:

(11) *If T is a triangulation of the polytope X, then the set X^k defined as the union of all simplexes of T with dimensions $\leqslant k$, is a homotopic (hence also a homological) k-skeleton of X.*

In fact, if X_0 is a closed subset of X and $\dim X_0 \leqslant k$ then, applying (10) to the inclusion $i \in X^{X_0}$, we infer that there exists a map $g \in X^{k X_0}$ homotopic to i. Consequently, $X_0 \underset{ht}{\leqslant} X^k$ and we infer that X^k is a homotopic k-skeleton of X.

Now let us prove the following

THEOREM. *Let X be an ANR-set with the property (Δ). Then*: 1^o *For every* $k = 0, 1, \ldots$ *there exists in X a homotopic k-skeleton* 2^o *If i is the inclusion map of a homological k-skeleton X^k of X, then the induced homomorphism i_* maps the group $H_k(X^k, \mathfrak{A})$ onto the group $H_k(X, \mathfrak{A})$.*

Proof. Let Q^ω denote the Hilbert cube, i.e. the subset of the Hilbert space consisting of all points $\{x_1, x_2, \ldots\}$ with $0 \leqslant x_n \leqslant 1/n$ for $n = 1, 2, \ldots$ We can assume that $X \subset Q^\omega$.

Let Q^n denote the n-dimensional Euclidean cube defined as the set of all points $\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \in Q^\omega$. Since X is an ANR-set, there exists a natural n and a polytope $W \subset Q^n$ such that X is a retract of the set Y consisting of all points $\{x_1, x_2, \ldots\} \in Q^\omega$ with $\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \in W$. Obviously, the polytope W is a deformation retract of Y.

Let r denote the retraction of Y to X. Consider a triangulation T of W and let W^k denote the union of all at most k-dimensional simplexes of T. It follows by (9) that for every $\varepsilon > 0$, there exists a map $s_k \in X^{W^k}$ such that

$$\dim s_k(W^k) \leqslant k, \tag{12}$$

$$\varrho(s_k(x), r(x)) < \varepsilon \text{ for every point } x \in W^k. \tag{13}$$

Let us prove that, for ε sufficiently small, the set $s_k(W^k)$ is a homotopic k-skeleton of X.

Since X is an ANR-set, we infer by (13) that, for ε sufficiently small, there exists an extension $s \in X^Y$ of s_k homotopic to r in X^Y. It means that there exists in X^Y a family of maps $\{\varphi_t\}$, dependent continuously on the parameter $0 \leqslant t \leqslant 1$, and such that

$$\varphi_0 = r \quad \text{and} \quad \varphi_1 = s. \tag{14}$$

Consider now a closed, at most k-dimensional, subset X_0 of X. Since Y is homotopically deformable over itself to the polytope W, the set X_0 can be homotopically carried over Y to a subset of W. Now, applying the lemma (10), we can extend this homotopy so that we obtain a deformation of X_0 into a subset W_0 of the set W^k. Thus, we see that there exists in Y_0^X a family $\{\psi_t\}$ of maps, continuously dependent on the parameter $0 \leqslant t \leqslant 1$, and satisfying the condition

$$\psi_0(x) = (x) \text{ and } \psi_1(x) \in W_0 \subset W^k \text{ for every } x \in X_0. \tag{15}$$

Now let us set

$$g_t(x) = \varphi_t \psi_t(x) \text{ for every } x \in X_0 \text{ and } 0 \leqslant t \leqslant 1.$$

Then $\{g_t\} \subset X^{X_0}$ is a family of maps, continuously dependent on the parameter $0 \leqslant t \leqslant 1$ and such that

$$g_0(x) = \varphi_0 \psi_0(x) = r(X) = x \text{ for every } x \in X_0,$$

$$g_1(x) = \varphi_1 \psi_1(x) \in s(W^k) = s_k(W^k) \text{ for every } x \in X_0.$$

Consequently, $\{g_t\}$ homotopically carries the set X_0 over X into a subset of the set $s_k(W^k)$. Hence, $s_k(W^k)$ is a homotopic k-skeleton of X and the first part of the Theorem is proved.

Passing to the second part, we keep the notations used in the proof of the first part. Consider the following maps:

$$W^k \xrightarrow[i_1]{} W \xrightarrow[i_2]{} Y \xrightarrow[s]{} X,$$

where i_1 and i_2 are inclusions. They induce the homomorphisms:

$$H_k(W^k, \mathfrak{A}) \xrightarrow[i_{1*}]{} H_k(W, \mathfrak{A}) \xrightarrow[i_{2*}]{} H_k(Y, \mathfrak{A}) \xrightarrow[s_*]{} H_k(X, \mathfrak{A})$$

and we see at once that i_{1*} is a homomorphism onto, i_{2*} is an isomorphism onto and — since the map s is homotopic to the retraction r — the homomorphism $s_* = r_*$ is also onto. If we observe that $s_k = si_2i_1$, we infer that

$$\operatorname{Im} s_{k*} = H_k(X, \mathfrak{A}).$$

But X^k is a homological k-skeleton of X and $\dim s_k(W^k) \leqslant k$. Consequently, if we denote by i_3 the inclusion of the set $s_k(W^k)$ into X, we infer that

$$H_k(X, \mathfrak{A}) = \operatorname{Im} s_{k*} = \operatorname{Im}(i_3 s_k)_* = \operatorname{Im}(i_{3*} s_{k*}) \subset \operatorname{Im} i_{3*} \subset \operatorname{Im} i_*.$$

Since $\operatorname{Im} i_* \subset H_k(X, \mathfrak{A})$, we get $\operatorname{Im} i_* = H_k(X, \mathfrak{A})$, and our proof is concluded.

6. Example. Let us show, on an example, that the second part of the last theorem ceases to be true for ANR-sets without property (Δ).

Let L be a simple arc lying on a 2-dimensional sphere S and let f be a continuous map of L onto a 3-dimensional cube Q^3, disjoint with S. If we match every point $x \in L$ with the point $f(x) \in Q^3$, we get from the set $S \cup Q^3$ a 3-dimensional space X. This space is an ANR-set ([2], p. 250) and its 2-dimensional group of Betti $H_2(X, \mathfrak{N})$ (where $\mathfrak{N}$ is the group of integers) is an infinite cyclic group. Moreover, it is clear that every closed proper subset X_1 of X is homotopically contractible in X to a point. It follows that the subset X^2 of X, consisting of only one point is a homotopic, hence also a homological 2-skeleton of X. If i denotes the inclusion of X^2 into X then $i_*(H_2(X^2, \mathfrak{N})) = (0)$ and consequently the image of i_* differs from the group $H_2(X, \mathfrak{N})$.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

On a Generalization of the Cohomotopy Groups

by

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Presented on July 14, 1960

In a recent note [1] I have introduced the notion of the homotopic k-skeleton of space X, defined as a closed subset X^k of X such that $\dim X^k \leqslant k$ and that every closed, at most k-dimensional subset X_0 of X can be carried into X^k by a continuous deformation over the space X. Moreover, it has been shown in [1] that if X is a polytope or, more generally, an ANR-set satisfying the condition Δ, then homotopic k-skeletons exist for every $k = 0, 1, \ldots$ Evidently, for arbitrary space X and $k \geqslant \dim X$, the set $X^k = X$ is a homotopic k-skeleton of X.

In the present note, I use the notion of the homotopic k-skeleton to assign to space X some groups $\pi_k^n(X)$ including, as a special case corresponding to $k = n$, the n-th cohomotopy group $\pi^n(X)$ of X.

1. Join of maps. Let X_0 be a closed subset of a compactum X and y_0 a point of a space Y, which we still suppose to be an ANR-set. Let $Y^X(X_0, y_0)$ denote the subset of the space Y^X of all continuous maps of X into Y, consisting of all maps f satisfying the inclusion

$$f(X_0) \subset (y_0).$$

The components of $Y^X(X_0, y_0)$ will be said to be *homotopy classes* in $Y^X(X_0, y_0)$. The homotopy class including $f \in Y^X(X_0, y_0)$ will be denoted by $[f, X_0, y_0]$ and the map f will be said to be a *representative* of this class. Instead of $[f, 0, y_0]$ we write $[f]$. The collection of all homotopy classes in $Y^X(X_0, y_0)$ will be denoted by $Y^X[X_0, y_0]$. Evidently, $Y^X(0, y_0) = Y^X$. Instead of $Y^X[0, y_0]$ we shall write $[Y^X]$.

Since Y is an ANR-set, a map g belongs to $[f, X_0, y_0]$ if and only if there exists a family of maps $\{\varphi_t\} \subset Y^X(X_0, y_0)$ depending continuously on the parameter $0 \leqslant t \leqslant 1$ and satisfying the condition:

$$\varphi_0 = f; \quad \varphi_1 = g.$$

Let $f_1, f_2 \in Y^X(X_0, y_0)$ and let us assume that there exist two open and disjoint subsets G_1, G_2 of X and two maps g_1, g_2 such that

$$g_1 \in [f_1, X_0, y_0); \quad g_2 \in [f_2, X_0, y_0],$$
$$g_1(X - G_1) \subset (y_0); \quad g_2(X - G_2) \subset (y_0).$$

Setting

$$f(x) = \begin{cases} y_0 \text{ for every } x \in X - G_1 - G_2 \\ f_1(x) \text{ for every } x \in G_1, \\ f_2(x) \text{ for every } x \in G_2, \end{cases}$$

we get a map $f \in Y^X(X_0, y_0)$, called *join* of the maps f_1 and f_2 (see [2], [6] and [3]). It is clear that if $f_1' \in [f_1, X_0, y_0]$ and $f_2' \in [f_2, X_0, y_0]$ then every join of the maps f_1 and f_2 is a join of maps f_1' and f_2'. Consequently, if we assign to the pair of homotopy classes $[f_1, X_0, y_0]$ and $[f_2, X_0, y_0]$ the set

$$J([f_1, X_0, y_0], [f_2, X_0, y_0]),$$

consisting of all homotopy classes $[f, X_0, y_0]$, where f is a *join* of f_1 and f_2, then we get the operation J (which we call the *operation of join*) for which arguments are some pairs of elements of the collection $Y^X[X_0, y_0]$ and the values are some subsets of this collection.

In some special cases the operation of join J is defined for every pair of elements of the collection $Y^X[X_0, y_0]$; it is so, for instance, of Y is the n-dimensional sphere and $\dim X < 2n$ (see [3]). Moreover, in some cases, the operation J is univalent (i.e. the set $J([f_1, X_0, y_0], [f_2, X_0, y_0])$ consists always of one homotopy class), and this gives the collection $Y^X[X_0, y_0]$ the structure of an Abelian group. Two following cases are especially important:

Case 1. X is the n-dimensional sphere and $\dim X < 2n - 1$.

In this case the collection $Y^X[X_0, y_0]$ with the operation of join becomes the n-th cohomotopy group $\pi^n(X, X_0)$ of the pair of spaces (X, X_0) (see [2] and [6]). The choice of the point y_0 is immaterial. In the case $X_0 = 0$, we get the n-th cohomotopy group $\pi^n(X)$ of space X.

Case 2. X is the n-dimensional sphere ($n > 1$), X_0 consists of one point x_0.

In this case the collection $Y^X[X_0, y_0]$ with the operation of join, becomes the n-th homotopy group $\pi_n(Y, y_0)$ of the space Y, introduced by Hurewicz (see [5]).

2. Group $\pi_k^n(X, A)$. Let A be a closed subset of X. Let us denote by $Y^{A \subset X}$ the subset of Y^A consisting of all maps $f \in Y^A$ extendable over X, i.e. such that there exists at least one extension $g \in Y^X$ of f. Since Y is an ANR-set, it is clear that all maps $f \in Y^A$, homotopic to a map $f \in Y^{A \subset X}$, belong to $Y^{A \subset X}$. It follows that $Y^{A \subset X}$ is the union of some homotopy classes belonging to $[Y^A]$. Let us denote by $[Y^{A \subset X}]$ the set consisting of all homotopy classes of maps belonging to $Y^{A \subset X}$. Hence,

$$[Y^{A \subset X}] \subset [Y^A].$$

Now let us assume that Y is the n-dimensional sphere $S = S^n$ and that

$$\dim A < 2n - 1.$$

Then we have the same situation as in Case 1 of No. **1**, and we infer that the operation of the join gives to the set $[Y^A] = [S^A]$ the structure of an Abelian group, in fact the cohomotopy group $\pi^n(A)$ of the set A.

The set $[S^A \subset X] \subset [Y^A]$ generates a subgroup of the group $[Y^A]$. Let us denote this subgroup by $\pi^n(A \subset X)$. Thus, the group $\pi^n(A \subset X)$ is defined for each closed subset A of X satisfying the condition $\dim A < 2n-1$. If $\dim X < 2n-1$ then we can set $A = X$ and we get the group $\pi^n(X \subset X)$ which coincides with the n-th cohomotopy group $\pi^n(X)$ of X.

3. An algebraic lemma. In the next paragraph we shall show that in the case when A is a homotopic k-skeleton of X and $k < 2n-1$, then the group $\pi^n(A \subset X)$ does not depend on the choice of the homotopic k-skeleton A. In the proof of this theorem we need the following simple algebraic

LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{L}$ be two Abelian groups and φ and ψ two homomorphisms:*

$$\varphi : \mathfrak{A} \to \mathfrak{L}; \quad \psi : \mathfrak{L} \to \mathfrak{A}.$$

Moreover, let P and Q be two subsets of $\mathfrak{A}$ and $\mathfrak{L}$, respectively, such that

$$\psi\varphi(x) = x \text{ for every } x \in P,$$

$$\varphi\psi(y) = y \text{ for every } y \in Q.$$

Let φ_P denote the partial homomorphism of φ, considered on the subgroup (P) of $\mathfrak{A}$, generated by the set P, and let ψ_Q denote the partial homomorphism of ψ, considered on the subgroup (Q) of $\mathfrak{L}$ generated by the set Q. Then φ_P is an isomorphism of (P) onto (Q), and ψ_Q is the isomorphism inverse to φ_P.

Proof. For every element x of (P) there exist elements $p_1, p_2, \ldots p_m$ of P and integral coefficients $n_1, n_2, \ldots, n_m$ such that

$$x = n_1 p_1 + n_2 p_2 + \ldots + n_m p_m.$$

It follows

$$\varphi_P(x) = \varphi(x) = n_1\varphi(p_1) + n_2\varphi(p_2) + \ldots + n_m\varphi(p_m)$$

and

$$\psi_Q\varphi_P(x) = \psi\varphi(x) = n_1\psi\varphi(p_1) + n_2\psi\varphi(p_2) + \ldots + n_m\psi\varphi(p_m) =$$
$$= n_1 p_1 + n_2 p_2 + \ldots + n_m p_m = x,$$

since $\psi\varphi(p_i) = p_i$ for every $i = 1, 2, \ldots, k$.

In a similar way we show that $\varphi_P\psi_Q(y) = y$ for every $y \in (Q)$.

4. Group $\pi_k^n(X)$. Keeping the notations of Nos. **1** and **2**, let us prove the following

THEOREM. *Let A and B be two homotopic k-skeletons of X and let S denote the n-dimensional sphere, where $k < 2n-1$. Then the group $\pi^n(A \subset X)$ is isomorphic to the group $\pi^n(B \subset X)$.*

Proof. Since A and B are homotopic k-skeletons of X, there exist two continuous maps

$$\alpha : A \to B \text{ and } \beta : B \to A$$

such that α is homotopic in X^A to the inclusion

$$i_A : A \to X,$$

and β is homotopic in X^B to the inclusion

$$i_B : B \to X.$$

Let us assign, to every map $g \in S^B$, the map $g\alpha \in S^A$. It is clear that, if $g, g' \in S^B$ are homotopic, then $g\alpha, g'\alpha \in S^A$ are also homotopic. It follows that this correspondence induces a transformation α_* of the set $[S^B]$ into the set $[S^A]$, $\alpha_* : [S^B] \to [S^A]$.

Moreover, if $g \in S^{B \subset X}$, i.e. if there exists an extension $\bar{g} \in S^X$ of g, then the map $g\alpha \in S^A$ is homotopic to the partial map $\bar{g}|A$. Consequently $g\alpha$ is extendable over X, i.e. $g\alpha \in S^{A \subset X}$. Thus we see that α_* maps $[S^{B \subset X}]$ into $[S^{A \subset X}]$.

Similarly, if we assign to every map $f \in S^A$ the map $f\beta \in S^B$, then we get the induced transformation β_* of the set $[S^A]$ into the set $[S^B]$, $\beta_* : [S^A] \to [S^B]$.

Moreover β_* maps $[S^{A \subset X}]$ into $[S^{B \subset X}]$.

Now let us fix an arbitrary point $y_0 \in S$ and consider the relation of the join J, by which the sets $[S^A]$ and $[S^B]$ become Abelian groups, in fact the n-th cohomotopy groups $\pi^n(A)$ and $\pi^n(B)$. Let us show that the transformations

$$\alpha_* : \pi^n(B) \to \pi^n(A) \text{ and } \beta_* : \pi^n(A) \to \pi^n(B)$$

are homomorphisms.

Consider two maps $g_1, g_2 \in S^B$. We can assume (replacing, if necessary, the given maps by some homotopic maps) that there exist in B two open and disjoint sets G_1 and G_2 such that

$$g_1(B - G_1) \subset (y_0) \text{ and } g_2(B - G_2) \subset (y_0).$$

Setting

$$g(x) = \begin{cases} y_0 \text{ for every } x \in B - G_1 - G_2, \\ g_1(x) \text{ for every } x \in G_1, \\ g_2(x) \text{ for every } x \in G_2, \end{cases}$$

we get a map $g \in S^B$ being a join of g_1 and g_2. Since G_1 and G_2 are open and disjoint, and the map $\alpha \in B^A$ is continuous, the counter-images $\alpha^{-1}(G_1)$ and $\alpha^{-1}(G_2)$ are open and disjoint subsets of A. Setting

$$f'(x) = \begin{cases} y_0 \text{ for } x \in A - \alpha^{-1}(G_1) - \alpha^{-1}(G_2), \\ g_1\alpha(x) \text{ for } x \in \alpha^{-1}(G_1), \\ g_2\alpha(x) \text{ for } x \in \alpha^{-1}(G_2), \end{cases}$$

we obtain a map $f' \in S^A$ being a join of maps $g_1\alpha, g_2\alpha \in S^A$. But $f' = g\alpha$ and we infer that, for every two elements $[g_1], [g_2] \in [S^B]$, the join of the elements $\alpha_*([g_1]) = [g_1\alpha]$, $\alpha_*([g_2]) = [g_2\alpha] \in [S^A]$ coincides with $\alpha_*([g]) = [g\alpha] \in [S^B]$.

Hence, α_* is a homomorphism.

By an analogous argument we show that β_* is a homomorphism.

In order to conclude the proof of the Theorem, it sufficies — by virtue of lemma of No. 3 — to prove that

(1) $$\alpha_* \beta_* ([f]) = [f] \text{ for every } [f] \in [S^{A \subset X}],$$

and

(2) $$\beta_* \alpha_* ([g]) = [g] \text{ for every } [g] \in [S^{B \subset X}].$$

Let $\{\varphi_t\} \subset X^A$ and $\{\psi_t\} \subset X^B$ be two families of maps, depending continuously on the parameter $0 \leqslant t \leqslant 1$ and such that:

$$\varphi_0(x) = x, \quad \varphi_1(x) = \alpha(x) \text{ for every } x \in A,$$
$$\psi_0(y) = y, \quad \psi_1(y) = \beta(y) \text{ for exery } y \in B.$$

Consider an extension $f \in S^X$ of the map $f \in S^{A \subset X}$ and an extension $g \in S^B$ of the map $g \in S^{B \subset X}$. Then the family $\{\bar{f}\varphi_t\} \subset S^A$ is a homotopy in S^A from $\bar{f}\varphi_0 = f$ to $\bar{f}\varphi_1 = f\alpha$, and the family $\{\bar{f}\psi_t\,\alpha\} \subset S^A$ is a homotopy in S^A from $\bar{f}\psi_0\,\alpha = \bar{f}\alpha$ to $f\psi_1\,\alpha = \bar{f}\beta\alpha$. Thus we have shown that the map f is homotopic in S^A to the map $f\beta\alpha$, i.e.

$$[f] = [f\beta\,\alpha] \text{ for every } [f] \in [S^{A \subset X}].$$

But, by the definition of α_* and β_* we have

$$\alpha_*\,\beta_*([f]) = \alpha([f\beta]) = [f\beta\,\alpha],$$

and thus relation (1) is proved.

In an analogous way we prove relation (2). Thus, the proof of the Theorem is concluded.

It follows from this Theorem that the algebraic structure of the group $\pi^n(A \subset X)$ does not depend on the choice of the homotopic k-skeleton A of X. For a given space X, it depends only on the number n (the dimension of the sphere S) and on the number k. The abstract group, isomorphic with all groups $\pi^n(A \subset X)$, where A is a homotopic k-skeleton of X (where $k < 2n - 1$), will be denoted by $\pi_k^n(X)$. For $k < n$ the groups $\pi_k^n(X)$ are manifestly trivial. From the remark at the end of No. **2** we see that in the case dim $X < 2n - 1$, the group $\pi^n_{\dim X}(X)$ coincides with the n-th cohomotopy group of X.

5. Final remarks. From the properties of the n-dimensional sphere S we have used, by the definition of the group $\pi_k^n(X)$ only the property that S is an ANR-set and that, for every space A, the operation of the join gives to $[S^A]$ the structure of an Abelian group. This gives an obvious opportunity to further generalization of the notion of the group $\pi_k^n(X)$.

Let us give some examples in which the structure of the group $\pi^n(X)$ is apparent:

Example 1. Let X be an m-dimensional sphere. Then the group $\pi_k^n(X)$ is trivial tor every $k < m$. In this case, every homotopic k-skeleton of X is homotopic in X fo a point, and consequently, the set $[S^A]$ contains only one element. In this case the hypothesis that $k < 2n - 1$ is superfluous.

Example 2. Let T_{2n} be the $2n$-dimensional manifold defined as the Cartesian product $S \times S$ of two n-dimensional spheres $S = S^n$. Let a be a fixed point of S. One sees easily that the set

$$A = (a) \times S \cup S \times (a)$$

is a homotopic k-skeleton of T_{2n}, for every $k = n,\ n+1, \ldots, 2n-1$. To group $\pi^n(A)$ is manifestly a free Abelian group generated by the homotopy classes $[f]$, $[g]$, where

$$f(x, y) = x \quad \text{and} \quad g(x, y) = y \quad \text{for every point} \quad (x, y) \in A.$$

The same formulas define the extensions f and g of f and g over T_{2n} and we conclude that $\pi^n(A \subset T_{2n}) = \pi^n(A)$ and consequently $\pi_k^n(T_{2n})$ is the free Abelian group of rank 2. Let us observe that, by a theorem of Hopf ([4], p. 436), the homotopic join of the maps $\bar{f}$ and $\bar{g}$ does not exist for even n, and consequently the cohomotopy group $\pi^n(T_{2n})$ is not defined.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATIK

Nuklearität gewisser Rellich-Sobolevschen Einbettungen. Anwendung auf Spektraltheorie der Differentialoperatoren

von

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Vorgelegt von A. MOSTOWSKI am 15. Juli 1960

Einführung

Die Wichtigkeit des Begriffes der Nuklearität der Abbildung $E \to F$ (E, F — Banach- oder allgemein lokalkonvexe Räume) wurde von A. Grothendick erkannt und ist das Hauptthema seiner grundlegenden Dissertation [3]. Dort sind auch einige klassisch Integralabbildungen in Bezug auf die Nuklearität untersucht.

In letzter Zeit wurde die Nuklearität der identischen Einbettung $E \to u \to Iu \in H$ als sehr wichtig für allgemeine Eigenfunktionentwicklungsätze erkannt [1], [2], [4]. Bekannt ist die Rolle der Sobolevschen Einbettungen $H^{[n/2]+1}(\Omega) \to C(\overline{\Omega})$ (Ω-beschränkt) in der Theorie der Randwertaufgaben partieller Differentialgleichungen. Diese Abbildungen (auch $H^{m+1} \to H^m$) sind für beschränkten Ω vollstetig. Da die Nuklearität eine viel stärkere Eigenschaft als die Vollstetigkeit ist, ist ihre Untersuchung entsprechend viel schwieriger und das ist der Grund, warum in konkreten Fällen noch recht wenig darüber bekannt ist.

In dieser Note wird die Nuklearität der Rellich—Sobolevschen Einbettungen $H_0^N(\Omega_N) \to H_0^k(\Omega_N)$ gezeigt. Als Anwendung wird mit Hilfe eines Satzes von einem der Verfasser [5] eine Verschärfung des Eigenfunktionsentwicklungssatzes von Berezanskiĭ—Gårding—Browder gewonnen.

1. Nuklearität der Einbettungen $H^{2m+k} \to H^k$

Es sei Ω_n ein beschränkter Bereich des Euklidischen Raumes E^n, $x = (x_1, \ldots, x_n) \in \Omega_n$ $C_0^\infty(\Omega_n)$ — die Menge der beliebig oft differenzierbaren (komplexen) Funktionen mit kompakten Trägern in Ω_n.

$$(u, v)_k = \sum_{|\alpha| \leqslant k} (D^\alpha u, D^\alpha v)_0,$$

wobei $(g, h)_0 = \int_{\Omega_n} ghdx$, $\|u\|_k^2 = (u, u)_k$. D^α — die Sobolev-Schwartzsche Ableitung der Ordnung $|\alpha| = \alpha_1 + \alpha_2 + \ldots \alpha_n$ ist.

$H_0^k = H_0^k(\Omega_n)$ bedeutet die Vervollständigung von $C_0^\infty(\Omega_n)$ in der $\| \ \|_k$-Norm.

$$\Delta^m u \overset{\text{df}}{=} \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha (D^\alpha u).$$

Die lineare Abbildung A zweier Banach-Räumen: $E \to F$ heisst nuklear, wenn sie in der folgender Form geschrieben werden kann:

$$E \ni u \to Au = \sum \langle u, e_i' \rangle f_i \in F,$$

wo $e_i' \in E'$ (d.h. e_i' sind stetige lineare Funktionale auf E) $f_i \in F$ und

$$\sum \|f_i'\|_F \|e_i'\|_{E'} < \infty.$$

Falls $E \ni u \to Iu = u \in F$, dann heisst I die (identische oder kanonische) Einbettung von E in F. Es gilt folgender

Satz 1. *Die Einbettung $H_0^{2m} \to H_0 = L^2(\Omega_n)$ ist nuklear falls $m \geqslant [n/2]+1$.*

Beweis. Aus dem Sobolevschen Lemma folgt, dass

(1) wenn $u \in H_0^m$, dann $u \in C(\overline{\Omega}_n)$

und

(2) $|u(x)| < C \|u\|_m$, C von u und x unabhängig.

Nach dem Satz von Fréchet — Riesz gilt es zu jedem x ein solches $k_x \in H_0^n$ (nach (1)) $k_x \in C(\overline{\Omega}_n)$, dass

(3) $$u(x) = (u, k_x)_m = \sum_{|\alpha| \leqslant m} (D^\alpha u, D^\alpha k_x).$$

Insbesondere haben wir für $u \in H_0^{2m}$ $(C_0^\infty \ni u_i \to u)$

$$u(x) = \lim_{i \to \infty} \left(\sum_{|\alpha| \leqslant m} (D^\alpha u_i, D^\alpha k_x)_0 = \lim_{i \to \infty} (\Delta^m u_i, k_x)_0 = (\Delta^m u, k_x)_0, \right.$$

wo

$$u_i \in C_0^\infty(\Omega_n); \ \|u - u_i\|_{2m} \to 0, \quad i \to \infty.$$

Jetzt setzen wir $k(x, y) = k_x(y)$ und haben also

$$u(x) = (\Delta^m u, k_x)_0 = \int \Delta_y^m \, u(y) \, k(x, y) \, dy$$

Aus (2) folgt, dass

(5) $$k(x, x) = \|k_x\|_m^2 = \left(\sup_u \frac{|(u, k_x)_m|}{\|u\|_m} \right)^2 = \left(\sup \frac{|u(x)|}{\|u\|_m} \right)^2 \leqslant C^2$$

(C von x und y unabhängig).

Es sei ψ_i^∞ eine orthonormale Basis in H_0^{2m}. Aus (3) haben wir

$$k_x = \sum_i (k_x, \psi_i)_m \psi_i = \sum_i \overline{(\psi_i, k_x)_m} \, \psi_i = \sum_i \overline{\psi_i(x)} \, \psi_i,$$

also

$$k(x, x) = \sum_i |\psi_i(x)|^2.$$

Folglich ist die Funktion $x \to k(x, x)$ messbar und wie aus (5) folgt

$$\int_{\Omega} k(x, x)\, dx < \infty. \tag{6}$$

Man kann die Einbettung $H_0^{2m} \to H_0$ aus der stetigen Abbildung

$$H_0^{2m} \ni u \to \Delta^m u \in H_0$$

und den Integraloperator (4) mit dem Kern $k(x, y)$

$$H_0 \in \Delta^m u \to u \in H_0$$

zusammensetzen. Dies folgt aus (4).

Der Integraloperator besitzt aber die Spur (6), er ist also nach einem Satze von Grothendick nuklear. Da aber die Superposition einer stetigen und nuklearen Abbildung nuklear ist, ist die Einbettung $H_0^{2m} \to H_0$ auch nuklear.

Aus dem Satz 1. folgt leicht der allgemeinere

Satz 2. *Die Einbettung* $H_0^{2m+2k} \to H_0^{2k}$ *ist nuklear für* $m = n/2,\ k > 0$.

Der Beweis folgt aus dem Diagram

$$H_0^{2m+2k} \ni u \to \Delta^k u \to \Delta^k u \to \Delta^{-k}(\Delta^k u) = u \in H_0^{2k}$$

$$H_0^{2m+k} \to H_0^{2m} \to H_0 \to H_0^{2k}.$$

Die erste der obigen Abbildungen ist stetig, die zweite (die Einbettung) gemäss dem Satze 1. ist nuklear; die dritte ist sogar isometrisch, da

$$\|u\|_{2k}^2 = (\Delta^{2k} u, u)_0 = (\Delta^k u, \Delta^k u)_0 = \|\Delta^k u\|_0^2,$$

woraus die Nuklearität folgt.

2. Eine Anwendung auf Spektraltheorie der Differentialoperatoren

In der Theorie der Differentialoperatoren spielt grosse Rolle der lokalkonvexe Raum

$$\mathfrak{H}^{2m}(\Omega) \overset{\text{df}}{=} \text{l. ind}_{p}\ H_0^{2m}(\Omega_n^p),$$

wo Ω — ein beliebiger n-dimensionaler Bereich $\subset E^n$ ist und $\Omega_n^p \nearrow \Omega$ eine aufsteigende Folge der beschränkter Unterbereiche von Ω. Diese Folge schöpft den Bereich Ω aus.

Für $H = L^2(\Omega)$, $\Phi = \mathfrak{H}^{2m}(\Omega)$ sind alle Voraussetzungen des Satzes 2. aus [5] erfüllt. Wir können also den folgenden Spektralsatz aussprechen (wegen Definitionen vgl. [4]):

SATZ. 3. *Es sei* $\varphi \in \mathfrak{H}^{2m}(\Omega)$. *Dann ist die verallgemeinerte Fouriertransformation durch die folgende Formel*

$$\Phi \ni \hat{\varphi} \to \varphi_k(\lambda) = (F\varphi)_k(\lambda) = \langle \varphi, e_k(\lambda) \rangle,$$

wo $e_k(\lambda) \in \mathfrak{H}^{2m} = (\mathfrak{H}^{2m}(\Omega))'$

$$k = 1, 2, \ldots \dim. \hat{H}(\lambda)$$

gegeben.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

MATHEMATICS

Distortion Theorems for Bounded Convex Functions

by

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Presented by K. KURATOWSKI on August 9, 1960

1. The Koebe constant for bounded convex functions

Let (C_M) denote the class of functions $f(z)$ regular and univalent in the unit circle $K = \{z : |z| < 1\}$, with $f(0) = 0$, $|f'(0)| = 1$, mapping the circle K onto a convex domain $f(K) = \Omega(f)$ contained in $K(M) = \{w : |w| < M\}$, $M > 1$. It is easy to prove that the boundary of $\Omega(f)$ is a simple closed convex Jordan curve $\Gamma(f)$ having the one-sided tangent everywhere. Besides, the set of points with different one-sided tangents is at most enumerable.

THEOREM 1. *Let $\delta(M)$ denote the Koebe constant for the class (C_M), i.e. let $\delta(M)$ be the radius of the largest disc whose centre lies in origin and which is contained in $\Omega(f)$ for every f in (C_M). Then*

$$\delta(M) = M \sin \Theta, \tag{1.1}$$

where Θ is the unique solution of the equation

$$(\pi+2\Theta) \sin \frac{4\pi\Theta}{\pi+2\Theta} = 2\pi M^{-1} \cos \Theta, \tag{1.2}$$

($M > 1$ being fixed) included in the open interval $(0, \pi/2)$. The extremal function f^* (for which the intersection $\Gamma(f) \cap \{w : |w| = \delta(M)\}$ is not empty) maps K onto $\Omega^* = K(M) \cap \{w : \operatorname{Re} w > -\delta(M)\}$ and is unique apart from rotations of K and Ω^* about the origin.

Proof. The class (C_M) is compact and this involves the existence of the extremal function f^*. We can suppose that $f^*(z) \neq -\delta(M)$ for all $z \in K$. Then, $\operatorname{Re} w > -\delta(M)$ for all $w \in f^*(K)$. Suppose, contrary to this, that $f^*(K)$ contains a point w' with $\operatorname{Re} w' \leqslant -\delta(M)$. Then, $f^*(K)$ being open contains w_1 such that $\operatorname{Re} w_1 < -\delta(M)$. By the convexity of $f^*(K)$, the ray of the straight line through w_1, $-\delta(M)$, which lies on the side of $-\delta(M)$ opposite to w_1, lies entirely outside $f^*(K)$. This ray contains, however, points of modulus less then $\delta(M)$ and this is contradictory to the extremal property of $\delta(M)$. Thus $f^*(K) \subset K(M) \cap \cap \{w : \operatorname{Re} w > -\delta(M)\} = \Omega^*$.

If the set $\Omega^* \setminus f^* K$ is not empty, then $r(0, \Omega^*) = \varrho > r(0, f^*(K)) = 1$ in view of the well known monotonity property of the inner conformal radius (for the definition and properties of the inner conformal radius $r(w, \Omega)$ of the domain Ω with respect to the point $w \in \Omega$ (see e.g. [2] p. 78). Let $f_1(z)$ map the circle K onto Ω^* so, that $f_1(0) = 0$. Then $|f_1'(0)| = \varrho > 1$, $f_1(K) \subset K(M)$, $f_1(z) \neq -\delta(M)$ for all $z \in K$. Obviously $\varrho^{-1} f_1 \in (C_M)$ and $\varrho^{-1} f_1(z) \neq -\varrho^{-1} \delta(M)$ for all $z \in K$. This gives a contradiction, since $\varrho^{-1} \delta(M) < \delta(M)$. We have proved simultaneously that $f^*(K) = \Omega^*$ for every extremal function f^* and this implies the uniqueness statement.

In order to obtain the numerical value of $\delta(M)$, we must find the function $f^*(z)$ mapping K onto Ω^* and such that $f^*(0) = 0, f^{*\prime}(0) = 1$. Since Ω^* is a circular wedge, such a mapping function may be found easily by standard methods (see e.g. [1], p. 76). The function

$$w = f^*(z, M) = \frac{M}{i} \frac{e^{-i\Theta} H(z) - e^{i\Theta}}{1 + H(z)}, \tag{1.3}$$

where

$$H(z) = e^{2i\Theta} \left(\frac{1 - \bar{\tau} z}{1 - \tau z}\right)^{1/\mu} = \left(\frac{\tau - z}{1 - z}\right)^{1/\mu}, \qquad \mu = \frac{2\pi}{\pi + 2\Theta}, \qquad \tau = e^{2i\Theta}, \tag{1.4}$$

maps conformally K onto the domain $\Omega(M, \Theta) = K(M) \cap \{\operatorname{Re} w > -M \sin \Theta\}$.

We have

$$\left(\frac{dw}{dz}\right)_{z=0} = r(0, \Omega(M, \Theta)) = \frac{M \sin 2\mu\Theta}{\mu \cos \Theta}.$$

In view of the monotonity property of the inner conformal radius, $r(0, \Omega(M, 0))$ is, $M > 1$ being fixed, a strictly monotonic function of $\Theta \in (0, \pi/2)$ which attains the value M $(M > 1)$ for $\Theta = \pi/2$ and tends monotonically to zero with decreasing Θ. This implies the existence of the unique solution of the equation

$$M \sin 2\mu\Theta / \mu \cos \Theta = 1 \tag{1.21}$$

which is equivalent to (1.2). Thus the Theorem 1 is proved.

2. The bounds for the modulus of $f(z)$ in (C_M)

Let $(\mathfrak{A})$ be the class of convex domains Ω containing the points $0, \eta$ (η is a fixed real number, $0 < \eta < M$) such that $\Omega \subset K(M)$, $r(0, \Omega) = 1$. Let $g(0, \eta, \Omega)$ denote the classical Green's function of Ω. We may confine ourselves to the classical Green's function, since the boundary of Ω is a simple Jordan curve.

If $\psi \in (C_M)$ and $\psi(re^{i\alpha}) = \eta$, then $\psi(K) = \Omega \in (\mathfrak{A})$ and, in view of the conformal invariance of the Green's function, we have

$$g(0, \eta, \Omega) = \log r^{-1}. \tag{2.1}$$

The equality (2.1) means that the problem of determining such a $\psi \in (C_M)$ that the equation $\psi(re^{i\alpha}) = \eta$ (with real α) is fulfilled with the least possible value of r, is equivalent to determining the extremal domain Ω_0 such that $\sup_{\Omega \in (\mathfrak{A})} g(0, \eta, \Omega) = g(0, \eta, \Omega_0)$. The extremal domain can be approximated by polygons. The

Hadamard's formulae for the variation of Green's function and of Robin's constant, combined with the method of Lagrange's multipliers give us the information of the shape of the domain Ω_n extremal in the class (Ω_n) The class $(\mathfrak{A}_n)$ is the subclass of $(\mathfrak{A})$ consisting of all polygons with at most n vertices.

LEMMA. *All the n angles of Ω_n are different from π and all its vertices apart from one at most, lie on the boundary of $K(M)$.*

Since Ω_0 may be approximated by a convergent subsequence $\{\Omega_{n_k}\}$ (the convergence is understood as the nucleus convergence) the Lemma helps us to determine the shape of Ω_0. It can be proved that $\Omega_0 = \Omega^*$ and thus Ω_0 does not depend on η. This means that the function $f^*(z, M)$ gives the upper bound for $|f|$ in (C_M). Similar considerations hold for minimizing $|f|$ and we obtain

THEOREM 2. *If $f \in (C_M)$, then*

$$-f^*(-|z|, M) \leqslant |f(z)| \leqslant f^*(|z|, M), \tag{2.2}$$

$f^*(z, M)$ being defined by (1.3) *and* (1.4) *with Θ being the unique solution of the Eq.* (1.2) *contained in* $(0, \pi/2)$.

The usual reasoning involving the validity of (2.2) for all $z \in K$ gives the precise inequality

$$|a_2| \leqslant A_2(M), \tag{2.3}$$

where

$$A_2(M) = 1/2\, f_z^{*\prime}(0, M) = M^{-1} \sin\Theta + \cos\frac{4\pi\Theta}{\pi+2\Theta}$$

with Θ defined as above.

The same method yields the bounds for the inner conformal radius $r(\eta, \Omega) = (1 - |z_0|^2)\ |\psi'(z_0)|$, if $\Omega \in (\mathfrak{A})$ and $\eta \in \Omega$ is fixed $[z_0 = \psi^{-1}(\eta),\ \psi \in (C_M)]$. These bounds can be derived from the fact that the maximal and also minimal (if $|\eta| < \delta(M)$) values of $r(\eta, \Omega)$, $\Omega \in (\mathfrak{A})$ η being fixed, are attained for $\Omega = \Omega^*$.

The detailed proofs will be published in the Ann. Univ. M. Curie-Skłodowska.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

On Line Strength Relations in the Doublet Arc Spectra of Aluminium, Gallium, Indium and Thallium

by

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Presented by W. RUBINOWICZ on July 6, 1960

Introduction

The quantum mechanical intensity formulae predict (in accord with the earlier known sum rules) the value 2:1 for line strength*) relations in (*S*—*P*)-doublets of one electron spectra.

However, departures from this value (named here briefly "departures from the sum rules") have been observed, first in the principal series of the alcalis. Fermi [6] gave a theoretical explanation, introducing into the intensity formulae radial functions modified by the first order peturbation caused by the spin-orbit interaction of the external electron. Similar effects, less systematically measured and theoretically not explained, exist in the spectra of many other elements. Among them, the doublet arc spectra of the third column elements (emitted when the valence *P*-electron is excited, the two *S*-electrons being in their ground states), seem to be of special interest: the relative intensities measured for Al, Ga, In and Tl (Ornstein, [15], Vonwiller, [22], Williams and Herlihy [24], Payne-Scott, [16], Voorhoeve, (23), Kunisz, [12], [13]) show a close analogy to those in the alcali doublets (in accordance with the general, far going similarity of both groups of spectra) with this difference only, that departures from the sum rules appear here also in sharp and diffuse series. This analogy suggests a simple possibility of generalizing the Fermi idea in order to explain these departures. Including the closed subshell of two *S*-electrons into the central field, one obtains a one-electron problem (with the $^2P_{1/2}$ ground state) as the zero order approximation. Let us try to predict the numerical values of the relative line strengths in the principal, sharp and diffuse series of the earth metal doublets, assuming their perturbation by the spin-orbit interaction of the valence *P*-electron. It is necessary for this purpose: (i) to extend the Fermi's perturbation calculus to the sharp and diffuse series, (ii) to adapt all three formulae to the case of a central field with the $^2P_{1/2}$ ground state.

*) The quantities introduced by Condon and Shortley [5] as a theoretical measure of the intensities.

Perturbation calculus for the sharp and diffuse series of one-electron spectra

Fermi assumed the spin-orbit interaction term of Thomas [20] and Frenkel [8]:

$$H'_F = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dU(r)}{dr}\vec{L}\cdot\frac{\hbar}{2}\vec{\sigma}, \qquad \text{for } l \neq 0, \tag{1}$$

$$H'_F = 0, \qquad \text{for } l = 0\,. \tag{1'}$$

$U(r)$ denoting the potential energy of the central field, m the electron mass, L its orbital angular momentum and $\vec{\sigma}$ the Pauli spin vector. Hence, he obtained the perturbed radial functions

$$R_{nlj} = R^0_{nl} + \begin{Bmatrix} l \\ -l(l+1) \end{Bmatrix} \frac{\hbar^2}{4m^2c^2} \sum_{k \neq n} \frac{\int_0^\infty \frac{1}{r}\frac{dU}{dr} R^0_{nl} R^0_{kl}\, r^2\, dr}{E^0_{nl} - E^0_{kl}} R^0_{kl} \tag{2}$$

$$\text{for } l \neq 0, \quad j = \begin{cases} l + 1/2 \\ l - 1/2 \end{cases}$$

$$R_{n0\,1/2} = R^0_{n0} \qquad \text{for } l = 0, \quad j = 1/2. \tag{2'}$$

(where R^0 are the zero order radial functions and E^0_{nl} the unperturbed energy levels) and finally, after some simple transformations and approximations, the expression for the relative line strengths in the principal series

$$\left(\frac{S_2}{S_1}\right)_n = 2\left(\frac{1+1/3\,F_n}{1-2/3\,F_n}\right)^2 \tag{3}$$

with

$$Fn = \sum_{k \neq n} \frac{\sqrt{\Delta_{n1}\cdot\Delta_{k1}}}{E^0_{n1} - E^0_{k1}} \frac{\varrho_{10;k1}}{\varrho_{10;n1}}, \tag{4}$$

$$\Delta_{nl} = E_{nl\,l+1/2} - E_{nl\,l-1/2}, \tag{5}$$

$$\varrho_{nl;n'l'} = \int_0^\infty R^0_{nl} R^0_{n'l'} r^3\, dr. \tag{6}$$

S_1, S_2 denote the line strengths corresponding to the upper $^2P_{1/2}$ and $^2P_{3/2}$ levels, respectively*). Owing to

$$\frac{\hbar^2}{4m^2c^2}\int_0^\infty \frac{1}{r}\frac{dU}{dr} R^0_{nl} R^0_{kl}\, r^2\, dr \approx \frac{1}{2l+1}\sqrt{\Delta_{nl}\cdot\Delta_{kl}} \qquad (\text{for } l \neq 0), \tag{7}$$

Fermi expressed the unknown integrals (7) by the experimentally given quantities Δ_{nl}.

The analogical effect in other series was never theoretically investigated. However, one cannot exclude it *a priori* assuming at the same time its existence in the principal

*) These symbols, different from those used by Fermi, are introduced here as more convenient for further calculations.

series. The similar calculations (based on (1), (1'), (2), (2'), and (7)) give for the sharp series

$$\frac{S_2}{S_1} = 2\left(\frac{1+1/3\,C_n}{1-2/3\,C_n}\right)^2, \tag{8}$$

$$C_n = \sum_{k \neq 2} \frac{\sqrt{\Delta_{21}\cdot\Delta_{k1}}}{E^0_{21}-E^0_{k1}} \frac{\varrho_{n0;\,k1}}{\varrho_{n0;\,21}}. \tag{9}$$

The symbols S_1, S_2 now refer to the components with lower levels $^2P_{1/2}$, $^2P_{3/2}$. Calculations for the diffuse series are a little longer, as both the upper and the lower levels are perturbed. Denoting line strengths of the three components: $(n^2\,D_{3/2} - 2^2\,P_{1/2})$, $n^2\,D_{3/2} - 2^2\,P_{3/2})$, $(n^2\,D_{5/2} - 2^2\,P_{3/2})$ by S_{I}, S_{II}, S_{III} successively, we obtain

$$S_{\mathrm{I}} : S_{\mathrm{II}} : S_{\mathrm{III}} = 5\,(1 - 2/3\,A_n - 3/5\,B_n)^2 : (1+1/3\,A_n - 3/5\,B_n)^2 : 9\,(1+ \\ +1/3\,A_n + 2/5\,B_n)^2 \tag{10}$$

with

$$A_n = \sum_{a \neq 2} \frac{\sqrt{\Delta_{21}\cdot\Delta_{a1}}}{E^0_{21}-E^0_{a1}} \frac{\varrho_{n2;\,a1}}{\varrho_{n2;\,21}} \tag{11}$$

$$B_n = \sum_{b \neq n} \frac{\sqrt{\Delta_{n2}\cdot\Delta_{b2}}}{E^0_{n2}-E^0_{b2}} \frac{\varrho_{b2;\,21}}{\varrho_{n2;\,21}} \tag{12}$$

For the unresolved D-levels one can put

$$S_{\mathrm{II}} + S_{\mathrm{III}} = S_2, \qquad S_{\mathrm{I}} = S_1, \tag{13}$$

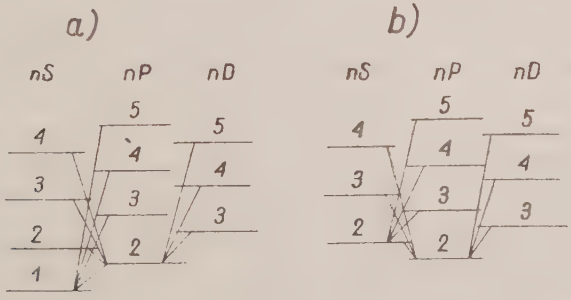
then obtaining

$$S_2 : S_1 = [9\,(1+1/3\,A_n + 2/5\,B_n)^2 + (1+1/3 A_n - 3/5\,B_n)^2] : 5\,(1 - 2/3\,A_n - 3/5\,B_n)^2. \tag{14}$$

(In the zero order approximation (10) and (14) reduce to the well known result: $S_{\mathrm{I}} : S_{\mathrm{II}} : S_{\mathrm{III}} = 5:1:9$, $S_2 : S_1 = 2:1$).

Numerical results for third column elements

The adaptation of (3), (8) and (14) to the alcali, or earth metal doublets consists in putting suitable values for n, k, a and b (as is evident from Fig. 1 and Table I; the conventional quantum numbers used here are for Al, Ga, In and Tl related



Qualitative diagram of the energy levels: *a*) for alcalis, *b*) for earth metal elements (doublet splitting omitted for simplicity)

to the true ones according to Table II). For the evaluation of (3), (8) and (14) E^0_{nl}, Δ_{nl} and $\varrho_{nl;n'l'}$ must be done. E^0_{nl} and Δ_{nl} are in general given by the spectroscopic data (f.i. Bacher and Goudsmith [3]). The main difficulty is inferred by the radial integrals ϱ. A large number of their values ought to be known (about fifty for each element) with a considerable accuracy, the numerical result of S_2/S_1 being extremely

TABLE I

Values of n, k, a and b in (4); (9), (11) and (12)

Series	Column	n	k	a	b	
Principal	I	2, 3, ...	2, 3, ...	—	—	$n \neq k$
	III	3, 4, ...				
Sharp	I	2, 3, ...	3, 4, ...	—	—	$k \neq 2$
	III					
Diffuse	I	3, 4, ...	—	3, 4, ...	3, 4, ...	$a \neq 2$, $b \neq n$
	III					

sensitive on even small changes in ϱ-values*). This difficulty, not entirely overcome even for better known case of the alcalis, is much more serious for the third column elements. The only procedure which seems possible to be applied is to use the tables of Bates and Damgaard [4]**). They offer a surprisingly simple method of obtaining a considerable number of ϱ-values, not less exact in general than they would be obtained by other, more laborious methods***). The effective quantum numbers necessary for the evaluation of ϱ's from the tables can easily be computed from the energy levels given by Bacher and Goudsmith [3]. The number of ϱ-values obtained in this way is still too small; therefore, results can be obtained for 3—4 lines of each series only, and their accuracy is not satisfactory, the worst one being for

*) This has been noticed by Sambursky [19].

**) The generalization of some particular ϱ-calculations, which were worked out in the early years of quantum mechanics for the first column elements, though of little intrinsic interest now, would probably be possible. However, it would be practically useless, as a procedure too tedious to be performed for the above mentioned large number of ϱ-values. They can also be expressed by some quantities known from the experiment — before all by the oscillator strengths f. In spite of considerable interest, attached to the f-values in the last years, (remarkable especially in astrophysical litterature), they have not been measured systematically enough for our purpose. Some particular values have been found by Kuhn [10], [11] Prokofiew, Soloview [17], Filippov, Prokofiew [7] and Allen and Asaad [1]. The detailed list of publications referring to ϱ or f-values for different elements is given by Unsold [21] and Regemorther [18]; other references, mainly to the astrophysical papers are cited by Aller [2]. Many known f-values are to be found in Landolt-Börnstein [14].

***) As shown by Bates and Damgaard [4]. The relative ϱ-values are still more accurate than the absolute ones.

TABLE II

Conventional (n) and true (n') principal quantum numbers for Al, Ga, In and Tl

Terms	n	n'			
		Al	Ga	In	Tl
S	2, 3, ...	4, 5, ...	5, 6, ...	6, 7, ...	7, 8, ...
P	2, 3, ...	3, 4, ...	4, 5, ...	5, 6, ...	6, 7, ...
D	3, 4, ...	3, 4, ...	4, 5, ...	5, 6, ...	6, 7, ...

the diffuse series. Some extrapolation procedures have been applied in order to reduce, as far as possible, the error issuing therefrom. The values of S_2/S_1 for Al, In and Tl are given in Table III *), together with the experimental results (cited after Kunisz [12]).

Discussion of results and final remarks

Results of the first order perturbation calculus seem admissible for the lighter elements; on the contrary, for Tl the correction assumes so large values that no exact results can be expected, except for the first (in the principal series still for the second) series member; this is a consequence of a very large separation of $P_{1/2,\,3/2}$ levels (7792.7 cm^{-1} for Tl, while it amounts 554 cm^{-1} for Cs). The good agreement between the theoretical and experimental results is to be noted in the principal series of In (the same may be supposed for Al and Ga). It suggests the correctness of the basic assumptions, underyling the foregoing considerations. The main qualitative difference between the alcali and earth metal spectra manifests itself in the first members of this series, where $S_2/S_2 < 2$ and $S_2/S_1 > 2$, respectively, both corrections, however, are too small to be observed.

In contrast to the satisfactory situation in the principal series, considerable discrepancies between experimental and theoretical values appear in the sharp and diffuse series, where even the sign of the correction and its general character is not confirmed by experimental data: the values $S_2/S_1 < 2$ predicted for all lines of these series have been observed for some first lines only. This difficulty is not restrained to the third column elements, as similar corrections in the sharp and diffuse series of the alcalis can be calculated by the same methods from (8) and (14), though the existence of such effects is not experimentally confirmed.

One may suspect the influence of the higher order corrections as well, as the presence of some other interactions not contained in (1), which partially cancel and even surpass the effects resulting from (1). These hypothetical corrections

*) The calculations for Ga are omitted since (i) the results would be overwhelmed with an additional error, caused by a relatively worse knowledge of energy levels for this element — before all of $3^2P_{1/2,\,3/2}$; (ii) the experimental data for comparison are missing till now, the only one is 1.55+0.07 for the first member of the sharp series, measured by Payne-Scott [16]. The approximate estimation of the theoretical results for this element is possible, owing to the regular change of corresponding values from one element to the other.

TABLE III

Theoretical and experimental values for S_2/S_1 in the principal, sharp and diffuse series of Al, In and Tl

Series	n	Al		In		Tl	
		experiment	theory	experiment	theory	experiment	theory
Principal	3		2.00		2.00		2.01
	4		2.04	2.52±0.3 (K)	2.95	4.5±0.44 (V) 4.4 (W)	6.45
	5		2.07	3.3 ±0.1 (K)	3.57	5.7±0.92 (V) 6.6±0.25 (W) 5.8±0.2 (K)	[11.7*]
	6		[2.12]	4.0 ±0.5 (K)	[3.61]	6.4 (V) 6.0±0.2 (W)	[17.5*]
	7			6.0 ±1.0 (K)		7±1 (V) 5.2±0.2 (W)	
Sharp	2	2.0±0.3 (V_0) 1.66±0.04 (K)	1.97	2.35±0.09 (P) 2.44±0.04 (K)	1.52	0.5 (O) 2.57±0.03 (V)	0.91
	3	2.0±0.3 (V_0) 2.03±0.03 (K)	1.82	3.33±0.05 (K)	0.33		0.02*
	4		1.64		0.04*		0.10*
	5		1.45		0.001*		[0.20*]
	6		[1.51]		[0.02*]		
Diffuse	3	1.9±0.3 (V_0) 1.67±0.05 (K)	[1.99]	2.73±0.10 (K)	[1.65]		[0.89]
	4	2.2±0.3 (V_0) 2.14±0.05 (K)	[1.90]	5.56±0.50 (K)	[1.02]		[0.10]
	5		[1.61]		[0.44]		[0.02*]

The abbreviations under the measured values denote: (O) — Ornstein, (V) — Vonwiller, (W) — Williams and Herlihy, (P) — Payne-Scott, (V_0) — Voorhoeve, (K) — Kunisz. The numbers in parentheses are partially based on extrapolated values of ϱ. The asterisks denote the results for which the perturbation calculus becomes dubious (the reasons being assigned in the following chapter).

(introduced by the perturbation calculus) should have some common features with the Fermi's correction: the increase with the atomic number of the element, and with the line number in the series, the larger values for the sharp and diffuse than for the principal series. Excluding, as less probable, the effects of different origin (f.i. the configuration interaction) one ought to take into account other relativistic effects and to revise at the same time the form of (1) (derived on the grounds of vector-model considerations). For a one-electron problem the Dirac theory gives (1) in the non-relativistic approximation together with two other terms

$$H_D' = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dU}{dr}\vec{L}\cdot\frac{\hbar}{2}\vec{\sigma} - \frac{1}{8m^3c^2}\vec{p}^4 - \frac{\hbar^2}{4m^2c^2}\frac{dU}{dr}\frac{d}{dr}. \tag{15}$$

These three terms give the correct result for the fine structure of the hydrogen lines; their influence on the eigenfunctions is then also probable. The possibility to extend some results of the Dirac theory to many electron problems (starting from the central field approximation) is discussed and confirmed by Hartree [9]. The cases of the alcali and earth metal atoms seem especially suitable for such treatment. Assuming (15) instead of (1) and performing the perturbation calculus one would obtain much more complicated expressions for S_2/S_1, containing other corrections besides F_n, C_n, A_n and B_n; these new terms would be always the same in the numerator and in the denominator of (3), (8) and (14), as two additional terms in (15) do not depend on the spin operator. Therefore, as long as all these corrections are small ($\ll 1$) so that the Taylor expansion of (3), (8) and (14) is admissible, additional terms would have to behave as those of higher order. On the contrary, for larger values of F_n, C_n, A_n and B_n (in particular approaching 3 or $^3/_2$) the "resonance effects" cause wild changes in S_2/S_1 (given by (3), (8), and (14)) which then become either very large, or very small. In this situation the role of additional terms would be decisive, even if they were much smaller than F_n, C_n, A_n and B_n *).

Obviously, only explicit calculations might prove (or disprove) the intuitive arguments quoted in this chapter. The problem is not easy to solve, since it is not possible to follow consequently the Fermi method of replacing the unknown integrals, difficult to calculate by some empirical quantities.

I would like to express my gratitude to Professor A. Jabłoński for suggesting this problem, and Dr. M. D. Kunisz for rendering accessible to me the results of her own measurements and other experimental data on this subject, prior to publication.

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*) The values with asterisks in Table III are just of this "resonance" character.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

Theory of Birefringence Induced in a Compressed Gas Mixture by an Electric Field Gradient

by

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Presented by W. RUBINOWICZ on July 6, 1960

Introduction

Quite recently, Buckingham [1] proposed a theory of optical birefringence induced in an isotropic medium by an electric field gradient. In the case of a gaseous medium, Buckingham derived the following expression for the difference

$$n_x - n_y = \frac{4\pi}{45} N_0 \left(B_{\alpha\beta:\alpha\beta} + \frac{1}{kT} \alpha_{\alpha\beta} \boldsymbol{H}_{\alpha\beta} \right) F_{xx} \tag{1}$$

between the refractive indexes of a gas, for light propagation in the direction of the axis of a four-wire condenser and with electric vectors in the x and y directions. N_0 is the number of molecules in unit volume, k — Boltzman's constant, T — absolute temperature and $F_{xx} = -F_{yy}$ — electric field gradient ($F_{zz} = 0$). $\boldsymbol{H}_{\alpha\beta}$ is the tensor of the quadrupole moment of the molecule, $\alpha_{\alpha\beta}$ — its polarizability tensor, whereas the tensor $B_{\alpha\beta:\gamma\delta}$ accounts for the quadrupole polarizability. The tensors $\alpha_{\alpha\beta}$ and $\boldsymbol{H}_{\alpha\beta}$ are symmetrical in the indexes α, β, and $B_{\alpha\beta:\gamma\delta}$ — in the pairs of indexes α, β and γ, δ. The tensors relate to the isolated molecule.

Buckingham proposed that Eq. (1) be used for the direct determination of the quadrupole moment of the molecule, which is given as a rule by

$$\boldsymbol{H}_{\alpha\beta} = \frac{1}{2} \sum_n e_n (3 r_{n\alpha} r_{n\beta} - r_n^2 \delta_{\alpha\beta}), \tag{2}$$

wherein e_n is the n-th electric charge of the molecule, and $\vec{r}_n$ — its radius vector. Summation extends over all the charges of the molecule. $\delta_{\alpha\beta}$ is the unit tensor.

Moreover, Buckingham [1] derived an expression for $n_x - n_y$ in dense media (gases at high pressure, or liquids). It is the aim of the present paper to apply this theory to multi-component systems, and, in particular, to compressed gas mixtures.

Theory for a multi-component system

The generalization of Buckingham's theory [1] for a multi-component system yields

$$\frac{n_x - n_y}{nF_{xx}} V_m = \frac{(n^2+2)^2\,(2\varepsilon+3)}{15\,n^2} Q_m, \tag{3}$$

wherein V_m is the molar volume of the system under consideration, n — refractive index and ε — dielectric permittivity. Q_m denotes the molecular constant of the optical birefringence induced in the multi-component system by the electric field gradient, and is given by

$$Q_m = \frac{4\pi}{45}\left\{\sum_{i=1}^{\nu}\left\langle\sum_{p=1}^{x_i N} B_{\alpha\beta:\alpha\beta}^{(p,i)}\right\rangle + \frac{1}{kT}\sum_{i=1}^{\nu}\sum_{j=1}^{\nu}\left\langle\sum_{p=1}^{x_i N}\sum_{q=1}^{x_j N}\alpha_{\alpha\beta}^{(p,i)} H_{\alpha\beta}^{(q,j)}\right\rangle\right\}, \tag{4}$$

x_i denotes the molar fraction of the i-th component of the system, ν — number of all components, and N — Avogadro's number, whilst the symbol $\langle\ \rangle$ is used to denote statistical averaging. The tensors $\alpha_{\alpha\beta}^{(p,i)}$, $H_{\alpha\beta}^{(p,i)}$ and $B_{\alpha\beta:\gamma\delta}^{(p,i)}$ now refer to the p-th molecule of species i, immersed in the medium.

We shall now apply Eq. (4) to a gas mixture by the method used by Buckingham and Raab [2] in their theory of the dielectric permittivity of a compressed gas mixture. To this purpose, the molecular constant Q_m is expanded in powers of $1/V_m$:

$$Q_m = A_Q + \frac{B_Q}{V_m} + \frac{C_Q}{V_m^2} + \ldots. \tag{5}$$

The expansion coefficients A_Q, B_Q, C_Q, etc. are termed the first, second, third, etc. virial coefficients of the molecular constant Q_m.

Discussion will be restricted to the virial coefficients A_Q and B_Q only; for a gas mixture, these are given by

$$A_Q = \sum_{i=1}^{\nu} x_i A_Q^{(i)}; \quad B_Q = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} x_i x_j B_Q^{(ij)}, \tag{6}$$

wherein, by the general expression of Eq. (4), we have

$$A_Q^{(i)} = \frac{4\pi}{45} N\left(B_{\alpha\beta:\alpha\beta}^{(i)} + \frac{1}{kT}\alpha_{\alpha\beta}^{(i)} H_{\alpha\beta}^{(i)}\right), \tag{7}$$

$$B_Q^{(ij)} = \frac{2\pi N^2}{45\Omega^2}\int\int\int\left\{B_{\alpha\beta:\alpha\beta}^{\binom{12}{ij}} - B_{\alpha\beta:\alpha\beta}^{(i)} - B_{\alpha\beta:\alpha\beta}^{(j)} + \right.$$

$$\left. + \frac{1}{kT}\left(\alpha_{\alpha\beta}^{\binom{12}{ij}} H_{\alpha\beta}^{\binom{12}{ij}} - \alpha_{\alpha\beta}^{(i)} H_{\alpha\beta}^{(i)} - \alpha_{\alpha\beta}^{(j)} H_{\alpha\beta}^{(j)}\right)\right\} e^{-\frac{u_{ij}^{(12)}}{kT}}\, d\vec{r}_{ij}^{(12)}\, d\omega_i^{(1)}\, d\omega_j^{(2)}. \tag{8}$$

Herein, the tensors $\alpha_{\alpha\beta}^{(i)}$, $H_{\alpha\beta}^{(i)}$ and $B_{\alpha\beta:\gamma\delta}^{(i)}$ are those of the isolated molecule of the i-th component of the gas, whilst $\alpha_{\alpha\beta}^{\binom{12}{ij}}$, $H_{\alpha\beta}^{\binom{(12)}{ij}}$ and $B_{\alpha\beta:\gamma\delta}^{\binom{12}{ij}}$ refer to the inter-

acting pair of molecules "1" and "2" belonging to the i-th and j-th gas component, respectively. The quantity $u_{ij}^{(12)}$ is the total potential energy of interaction between a pair of molecules belonging to the i-th and j-th species; $r_{ij}^{(12)}$ denotes the distance separating their centres, and $\omega_i^{(1)}$ and $\omega_j^{(2)}$ — their respective orientational variables; $\Omega = \int d\omega_i^{(1)} = \int d\omega_j^{(2)}$.

If, in particular, the tensors $\alpha_{\alpha\beta}^{\binom{12}{ij}}$, $H_{\alpha\beta}^{\binom{12}{ij}}$ and $B_{\alpha\beta:\gamma\delta}^{\binom{12}{ij}}$ are independent of the molecular fields, the second virial coefficient given by Eq. (8) reduces to the simpler form

$$(9) \qquad B_Q^{(ij)} = \frac{2\pi N^2}{45kT\Omega^2} \int\int\int \left(\alpha_{\alpha\beta}^{(1,i)} H_{\alpha\beta}^{(2,j)} + H_{\alpha\beta}^{(1,i)} \alpha_{\alpha\beta}^{(2,j)}\right) e^{-\frac{u_{ij}^{(12)}}{kT}} d\vec{r}_{ij}^{(12)}\, d\omega_i^{(1)}\, d\omega_j^{(2)}.$$

The foregoing expression will now be discussed separately for the case of a mixture of non-dipolar gases and for that of polar ones.

Non-dipolar gas mixture

The total potential energy of the interaction between a pair of non-dipolar molecules of the i-th and j-th gas components, respectively, possessing a quadrupole moment and presenting anisotropic polarizability, is given by the expression [3]

$$(10) \qquad u_{ij}^{(12)} = u(r_{ij}) - \frac{1}{9} H_{\alpha\beta}^{(1,i)} H_{\gamma\delta}^{(2,j)} T_{\alpha\beta\gamma\delta}^{(ij)} - \\ - \frac{1}{18} \left(\alpha_{\alpha\beta}^{(1,i)} H_{\gamma\delta}^{(2,j)} H_{\varepsilon\eta}^{(2,j)} + \alpha_{\alpha\beta}^{(2,j)} H_{\gamma\delta}^{(1,i)} H_{\varepsilon\eta}^{(1,i)}\right) T_{\alpha\gamma\delta}^{(ij)} T_{\beta\varepsilon\eta}^{(ij)},$$

wherein $u(r_{ij})$ denotes the energy of central interaction of the molecules, the second term accounts for the quadrupole-quadrupole interaction energy, and the third — for that of quadrupole-induced dipole interaction. The tensors $T_{\alpha\beta\gamma}^{(ij)}$ and $T_{\alpha\beta\gamma\delta}^{(ij)}$ characterizing dipole-quadrupole (r_{ij}^{-4}) and quadrupole-quadrupole (r_{ij}^{-5}) interactions have been given here in the form in which they occur in Pople's paper [4].

Considering that the tensorial molecular interaction energy in Eq. (10) represents a perturbation in $u(r_{ij})$, Eq. (9) yields, by the method used in the theory of light scattering virial coefficients [3],

$$(11) \qquad B_Q^{(ij)} = {}_{\text{quadr-quadr}}B_Q^{(ij)} + {}_{\text{quadr-ind. dip}}B_Q^{(ij)},$$

wherein

$$(12) \qquad {}_{\text{quadr-quadr}}B_Q^{(ij)} = \frac{64\pi N^2}{14175\, k^3T^3} \{(3\alpha_{\alpha\beta}^{(i)} H_{\alpha\gamma}^{(i)} H_{\beta\gamma}^{(i)} - \alpha_{\alpha\alpha}^{(i)} H_{\beta\gamma}^{(i)} H_{\beta\gamma}^{(i)}) H_{\delta\varepsilon}^{(j)} H_{\delta\eta}^{(j)} H_{\varepsilon\eta}^{(j)} + \\ + (3\alpha_{\alpha\beta}^{(j)} H_{\alpha\gamma}^{(j)} H_{\beta\gamma}^{(j)} - \alpha_{\alpha\alpha}^{(j)} H_{\beta\gamma}^{(j)} H_{\beta\gamma}^{(j)}) H_{\delta\varepsilon}^{(i)} H_{\delta\eta}^{(i)} H_{\varepsilon\eta}^{(i)}\} \langle r_{ij}^{-10} \rangle + \ldots,$$

is the second virial coefficient accounting for quadrupole-quadrupole interaction, whereas

$$(13) \qquad {}_{\text{quadr-ind. dip}}B_Q^{(ij)} = \frac{4\pi N^2}{4725\, k^2T^2} \{(3\alpha_{\alpha\beta}^{(i)} \alpha_{\alpha\beta}^{(i)} - \alpha_{\alpha\alpha}^{(i)} \alpha_{\beta\beta}^{(i)}) H_{\gamma\delta}^{(j)} H_{\gamma\varepsilon}^{(j)} H_{\delta\varepsilon}^{(j)} + \\ + (3\alpha_{\alpha\beta}^{(j)} \alpha_{\alpha\beta}^{(j)} - \alpha_{\alpha\alpha}^{(j)} \alpha_{\beta\beta}^{(j)}) H_{\gamma\delta}^{(i)} H_{\gamma\varepsilon}^{(i)} H_{\delta\varepsilon}^{(i)} + (3\alpha_{\alpha\beta}^{(i)} H_{\alpha\gamma}^{(i)} H_{\beta\gamma}^{(i)} - \\ - \alpha_{\alpha\alpha}^{(i)} H_{\beta\gamma}^{(i)} H_{\beta\gamma}^{(i)}) \alpha_{\delta\varepsilon}^{(j)} H_{\delta\varepsilon}^{(j)} + (3\alpha_{\alpha\beta}^{(j)} H_{\alpha\gamma}^{(j)} H_{\beta\gamma}^{(j)} - \alpha_{\alpha\alpha}^{(j)} H_{\beta\gamma}^{(j)} H_{\beta\gamma}^{(j)}) \alpha_{\delta\varepsilon}^{(i)} H_{\delta\varepsilon}^{(i)}\} \langle r_{ij}^{-8} \rangle + \ldots$$

accounts for quadrupole-induced dipole interactions, and

$$\langle r_{ij}^{-n} \rangle = \int_0^\infty r_{ij}^{-n} e^{-\frac{u(r_{ij})}{kT}} dr_{ij} \tag{14}$$

is a mean value depending solely on the model of central forces $u(r_{ij})$ adopted.

Assuming the Lennard-Jones potential [5] for $u(r_{ij})$, in the form [6]

$$u(r_{ij}) = 4\varepsilon_{ij}\left\{\left(\frac{\sigma_{ij}}{r_{ij}}\right)^{12} - \left(\frac{\sigma_{ij}}{r_{ij}}\right)^{6}\right\}, \tag{15}$$

the expression of Eq. (14) yields [7]

$$\langle r_{ij}^{-n} \rangle = \frac{\pi \sigma_{ij}^{3-n}}{3 y_{ij}^4} H_n(y_{ij}); \qquad y_{ij} = 2\left(\frac{\varepsilon_{ij}}{kT}\right)^{1/2}. \tag{16}$$

Herein, ε_{ij} and σ_{ij} are the well-known central forces parameters having the dimensions of energy and length, and

$$H_n(y_{ij}) = y_{ij}^{\frac{27-n}{6}} \sum_{s=0}^{\infty} \Gamma\left(\frac{6s+n-3}{12}\right) \frac{y_{ij}^s}{s!} \tag{17}$$

— the functions introduced by Pople [8]; these have been tabulated by Buckingham and Pople [7] for n ranging from 6 to 17 and for values $0.6 \leqslant y \leqslant 3.2$.

For molecules possessing axial symmetry, Eqs. (12) and (13), with respect to (16), reduce to

$$_{\text{quadr-quadr}}B_Q^{(ij)} = \frac{\pi^2 N^2}{3150\, kT\, \sigma_{ij}^7 \varepsilon_{ij}^2} (\alpha_{(i)}\, \delta_{(i)}\, \boldsymbol{H}_{(j)} + \boldsymbol{H}_{(i)}\, \alpha_{(j)}\, \delta_{(j)})\, \boldsymbol{H}_{(i)}^2\, \boldsymbol{H}_{(j)}^2\, H_{10}(y_{ij}) + \dots, \tag{18}$$

$$_{\text{quadr-ind. dip}}B_Q^{(ij)} = \frac{\pi^2 N^2}{1050\, kT\, \sigma_{ij}^5 \varepsilon_{ij}\, y_{ij}^2} (\alpha_{(i)}\, \delta_{(i)}\, \boldsymbol{H}_{(j)} + \boldsymbol{H}_{(i)}\, \alpha_{(j)}\, \delta_{(j)})\, (\alpha_{(i)}\, \delta_{(i)}\, \boldsymbol{H}_{(j)}^2 + \boldsymbol{H}_{(i)}^2\, \alpha_{(j)}\, \delta_{(j)})\, H_8(y_{ij}) + \dots, \tag{19}$$

wherein

$$\boldsymbol{H} = \boldsymbol{H}_{33} = \frac{1}{2} \sum_n e_n (3Z_n^2 - r_n^2) = -2\boldsymbol{H}_{11} = -2\boldsymbol{H}_{22} \tag{20}$$

is the quadrupole moment of an axially symmetric molecule, and the quantities

$$\delta = \frac{\alpha_{\parallel} - \alpha_{\perp}}{3\alpha} \quad \text{and} \quad \alpha = \frac{1}{3}(\alpha_{\parallel} + 2\alpha_{\perp}) \tag{21}$$

denote the optical anisotropy of the molecule and its mean polarizability, respectively, $\alpha_{\parallel}$ being the polarizability in the direction of the axis of symmetry of the molecule, and $\alpha_{\perp}$ — that in the direction perpendicular to this axis.

Polar gas mixture

For polar molecules possessing a permanent dipole moment μ_α and quadrupole moment $H_{\alpha\beta}$, the total potential interaction energy is given by (see [3], [4])

$$(22)\quad u_{ij}^{(12)} = u(r_{ij}) + \mu_\alpha^{(1,i)}\mu_\beta^{(2,j)}T_{\alpha\beta}^{(ij)} + \frac{1}{3}(\mu_\alpha^{(1,i)}H_{\beta\gamma}^{(2,j)} - H_{\alpha\beta}^{(1,i)}\mu_\gamma^{(2,j)})T_{\alpha\beta\gamma}^{(ij)} -$$

$$- \frac{1}{9}H_{\alpha\beta}^{(1,i)}H_{\gamma\delta}^{(2,j)}T_{\alpha\beta\gamma\delta}^{(ij)} - \frac{1}{2}(\alpha_{\alpha\beta}^{(1,i)}\mu_\gamma^{(2,j)}\mu_\delta^{(2,j)} + \alpha_{\alpha\beta}^{(2,j)}\mu_\gamma^{(1,i)}\mu_\delta^{(1,i)})T_{\alpha\gamma}^{(ij)}T_{\beta\delta}^{(ij)} + \ldots;$$

here, in addition to the energy $u(r_{ij})$ due to central forces, terms accounting for the energy of electrostatic dipole-dipole (r_{ij}^{-3}), dipole-quadrupole (r_{ij}^{-4}), quadrupole-quadrupole (r_{ij}^{-5}), dipole-induced dipole (r_{ij}^{-6}) etc. interactions occur.

Substitution of the foregoing expansion in Eq. (9) yields the second virial coefficient in the form

$$(23)\quad B_Q^{(ij)} = {}_{\text{dip-dip}}B_Q^{(ij)} + {}_{\text{dip-quadr}}B_Q^{(ij)} + {}_{\text{quadr-quadr}}B_Q^{(ij)} + {}_{\text{dip-ind.dip}}B_Q^{(ij)} + \ldots$$

The first term herein accounts for dipole-dipole interaction and is given by

$$(24)\quad {}_{\text{dip-dip}}B_Q^{(ij)} = \frac{\pi N^2}{3375k^3T^3}\left\{\left(3\alpha_{\alpha\beta}^{(i)}\mu_\alpha^{(i)}\mu_\beta^{(i)} - \alpha_{\alpha\alpha}^{(i)}\mu_\beta^{(i)}\mu_\beta^{(i)}\right)H_{\gamma\delta}^{(j)}\mu_\gamma^{(j)}\mu_\delta^{(j)} + \right.$$

$$\left. + \left(3\alpha_{\alpha\beta}^{(j)}\mu_\alpha^{(j)}\mu_\beta^{(j)} - \alpha_{\alpha\alpha}^{(j)}\mu_\beta^{(j)}\mu_\beta^{(j)}\right)H_{\gamma\delta}^{(i)}\mu_\gamma^{(i)}\mu_\delta^{(i)}\right\}\left\{\langle r_{ij}^{-6}\rangle + \frac{12\mu_{(i)}^2\mu_{(j)}^2}{49k^2T^2}\langle r_{ij}^{-12}\rangle + \ldots\right\},$$

whilst the second term, which relates to dipole-quadrupole interaction, has the form

$$(25)\quad {}_{\text{dip-quadr}}B_Q^{(ij)} = \frac{4\pi N^2}{4725k^3T^3}\left\{\left[\left(3\alpha_{\alpha\beta}^{(i)}\mu_\alpha^{(i)}\mu_\beta^{(i)} - \alpha_{\alpha\alpha}^{(i)}\mu_\beta^{(i)}\mu_\beta^{(i)}\right)H_{\gamma\delta}^{(j)}H_{\gamma\varepsilon}^{(j)}H_{\delta\varepsilon}^{(j)} + \right.\right.$$

$$+ \left(3\alpha_{\alpha\beta}^{(j)}\mu_\alpha^{(j)}\mu_\beta^{(j)} - \alpha_{\alpha\alpha}^{(j)}\mu_\beta^{(j)}\mu_\beta^{(j)}\right)H_{\gamma\delta}^{(i)}H_{\gamma\varepsilon}^{(i)}H_{\delta\varepsilon}^{(i)} + \left(3\alpha_{\alpha\beta}^{(i)}H_{\alpha\gamma}^{(i)}H_{\beta\gamma}^{(i)} - \right.$$

$$\left. - \alpha_{\alpha\alpha}^{(i)}H_{\beta\gamma}^{(i)}H_{\beta\gamma}^{(i)}\right)H_{\delta\varepsilon}^{(j)}\mu_\delta^{(j)}\mu_\varepsilon^{(j)} + \left(3\alpha_{\alpha\beta}^{(j)}H_{\alpha\gamma}^{(j)}H_{\beta\gamma}^{(j)} - \alpha_{\alpha\alpha}^{(j)}H_{\beta\gamma}^{(j)}H_{\beta\gamma}^{(j)}\right)H_{\delta\varepsilon}^{(i)}\mu_\delta^{(i)}\mu_\varepsilon^{(i)}\left.\right]\langle r_{ij}^{-8}\rangle -$$

$$- \frac{16}{15kT}\left[\left(3\alpha_{\alpha\beta}^{(i)}H_{\alpha\gamma}^{(i)}\mu_\beta^{(i)}\mu_\gamma^{(i)} - \alpha_{\alpha\alpha}^{(i)}H_{\beta\gamma}^{(i)}\mu_\beta^{(i)}\mu_\gamma^{(i)} - \alpha_{\alpha\beta}^{(i)}H_{\alpha\beta}^{(i)}\mu_\gamma^{(i)}\mu_\gamma^{(i)}\right)\left(3H_{\delta\varepsilon}^{(j)}H_{\eta\delta}^{(j)}\mu_\varepsilon^{(j)}\mu_\eta^{(j)} - \right.\right.$$

$$\left. - H_{\delta\varepsilon}^{(j)}H_{\delta\varepsilon}^{(j)}\mu_\eta^{(j)}\mu_\eta^{(j)}\right) + \left(3\alpha_{\alpha\beta}^{(j)}H_{\alpha\gamma}^{(j)}\mu_\beta^{(j)}\mu_\gamma^{(j)} - \alpha_{\alpha\alpha}^{(j)}H_{\beta\gamma}^{(j)}\mu_\beta^{(j)}\mu_\gamma^{(j)} - \right.$$

$$\left.\left. - \alpha_{\alpha\beta}^{(j)}H_{\alpha\beta}^{(j)}\mu_\gamma^{(j)}\mu_\gamma^{(j)}\right)\left(3H_{\delta\varepsilon}^{(i)}H_{\delta\eta}^{(i)}\mu_\varepsilon^{(i)}\mu_\eta^{(i)} - H_{\delta\varepsilon}^{(i)}H_{\delta\varepsilon}^{(i)}\mu_\eta^{(i)}\mu_\eta^{(i)}\right)\right]\langle r_{ij}^{-11}\rangle -$$

$$- \frac{4}{15kT}\left[5\left(\alpha_{\alpha\beta}^{(i)}H_{\alpha\beta}^{(i)}H_{\gamma\delta}^{(j)}H_{\gamma\delta}^{(j)} + \alpha_{\alpha\beta}^{(j)}H_{\alpha\beta}^{(j)}H_{\gamma\delta}^{(i)}H_{\gamma\delta}^{(i)}\right)\mu_{(i)}^2\mu_{(j)}^2 + 2\left(3H_{\alpha\beta}^{(i)}H_{\alpha\gamma}^{(i)}\mu_{\alpha\beta}^{(i)}\mu_\gamma^{(i)} - \right.\right.$$

$$- H_{\alpha\beta}^{(i)}H_{\alpha\beta}^{(i)}\mu_{(i)}^2\left)\alpha_{\delta\varepsilon}^{(j)}H_{\delta\varepsilon}^{(j)}\mu_{(j)}^2 + 2\left(3H_{\alpha\beta}^{(j)}H_{\alpha\gamma}^{(j)}\mu_\beta^{(j)}\mu_\gamma^{(j)} - H_{\alpha\beta}^{(j)}H_{\alpha\beta}^{(j)}\mu_{(j)}^2\right)\alpha_{\delta\varepsilon}^{(i)}H_{\delta\varepsilon}^{(i)}\mu_{(i)}^2 + \right.$$

$$+ 2\left(3\alpha_{\alpha\beta}^{(i)}H_{\alpha\gamma}^{(i)}\mu_\beta^{(i)}\mu_\gamma^{(i)} - \alpha_{\alpha\alpha}^{(i)}H_{\beta\gamma}^{(i)}\mu_\beta^{(i)}\mu_\gamma^{(i)}\right)H_{\delta\varepsilon}^{(j)}H_{\delta\varepsilon}^{(j)}\mu_{(j)}^2 + 2\left(3\alpha_{\alpha\beta}^{(j)}H_{\alpha\gamma}^{(j)}\mu_{\beta\beta}^{(j)}\mu_\gamma^{(j)} - \right.$$

$$\left.\left.\left. - \alpha_{\alpha\alpha}^{(j)}H_{\beta\gamma}^{(j)}\mu_\beta^{(j)}\mu_\gamma^{(j)}\right)H_{\delta\varepsilon}^{(i)}H_{\delta\varepsilon}^{(i)}\mu_{(i)}^2\right]\langle r_{ij}^{-11}\rangle + \ldots\right\}.$$

The quantity $_{\text{quadr-quadr.}} B_Q^{(ij)}$ is given by Eq. (12), whereas the contribution from quadr-quadr. dipole interaction has the form

$$(26)\quad {}_{\text{dip-ind. dip}} B_Q^{(ij)} = \frac{\pi N^2}{3375\, k^2 T^2} \{ (3\, \alpha_{\alpha\beta}^{(i)} \alpha_{\alpha\beta}^{(i)} - \alpha_{\alpha\alpha}^{(i)} \alpha_{\beta\beta}^{(i)}) \boldsymbol{H}_{\gamma\delta}^{(j)} \mu_\gamma^{(j)} \mu_\delta^{(j)} +$$
$$+ (3\, \alpha_{\alpha\beta}^{(j)} \alpha_{\alpha\beta}^{(j)} - \alpha_{\alpha\alpha}^{(j)} \alpha_{\beta\beta}^{(j)}) \boldsymbol{H}_{\gamma\delta}^{(i)} \mu_\gamma^{(i)} \mu_\delta^{(i)} + (3\, \alpha_{\alpha\beta}^{(i)} \mu_\alpha^{(i)} \mu_\beta^{(i)} - \alpha_{\alpha\alpha}^{(i)} \mu_\beta^{(i)} \mu_\beta^{(i)}) \boldsymbol{H}_{\gamma\delta}^{(j)} \alpha_{\gamma\delta}^{(j)} +$$
$$+ (3\, \alpha_{\alpha\beta}^{(j)} \mu_\alpha^{(j)} \mu_\beta^{(j)} - \alpha_{\alpha\alpha}^{(j)} \mu_\beta^{(j)} \mu_\beta^{(j)}) \boldsymbol{H}_{\gamma\delta}^{(i)} \alpha_{\gamma\delta}^{(i)} \} \langle r_{ij}^{-6} \rangle + \dots .$$

For axially symmetric molecules, with respect to Eq. (16), the foregoing expressions reduce to

$$(27)\quad {}_{\text{dip-dip}} B_Q^{(ij)} = \frac{\pi^2 N^2}{27000\, kT \sigma_{ij}^3 \varepsilon_{ij}^2} \{ \alpha_{(i)} \delta_{(i)} \mu_{(i)}^2 \boldsymbol{H}_{(j)} \mu_{(j)}^2 + \boldsymbol{H}_{(i)} \mu_{(i)}^2 \alpha_{(j)} \delta_{(j)} \mu_{(j)}^2 \} \times$$
$$\times \left\{ H_6(y_{ij}) + \frac{3\, \mu_{(i)}^2 \mu_{(j)}^2 y_{ij}^4}{196\, \sigma_{ij}^6 \varepsilon_{ij}^2} H_{12}(y_{ij}) + \dots \right\},$$

$$(28)\quad {}_{\text{dip-quadr}} B_Q^{(ij)} = \frac{\pi^2 N^2}{12600\, kT \sigma_{ij}^5 \varepsilon_{ij}^2} \{ \alpha_{(i)} \delta_{(i)} \boldsymbol{H}_{(j)} + \boldsymbol{H}_{(i)} \alpha_{(j)} \delta_{(j)} \} \times$$
$$\times \left\{ (\mu_{(i)}^2 \boldsymbol{H}_{(j)}^2 + \boldsymbol{H}_{(i)}^2 \mu_{(j)}^2) H_8(y_{ij}) - \frac{y_{ij}^2}{\sigma_{ij}^3 \varepsilon_{ij}} \mu_{(i)}^2 \boldsymbol{H}_{(i)} \boldsymbol{H}_{(j)} \mu_{(j)}^2 H_{11}(y_{ij}) + \dots \right\},$$

$$(29)\quad {}_{\text{dip-ind. dip}} B_Q^{(ij)} = \frac{\pi^2 N^2}{2250\, kT \sigma_{ij}^3 \varepsilon_{ij} y_{ij}^2} \{ \alpha_{(i)} \delta_{(i)} \boldsymbol{H}_{(j)} + \boldsymbol{H}_{(i)} \alpha_{(j)} \delta_{(j)} \} \times$$
$$\times \{ \alpha_{(i)} \delta_{(i)} \mu_{(j)}^2 + \mu_{(i)}^2 \alpha_{(j)} \delta_{(j)} \} H_6(y_{ij}) + \dots .$$

Binary gas mixture

For a binary gas mixture, the expansion of Eq. (5) can be put in the following form:

$$(30)\quad Q_m = x_1 A_Q^{(1)} + x_2 A_Q^{(2)} + \frac{1}{V_m} (x_1^2 B_Q^{(11)} + 2 x_1 x_2 B_Q^{(12)} + x_2^2 B_Q^{(22)}) + \dots ,$$

with

$$(31)\quad x_1 + x_2 = 1, \qquad V_m = \frac{1}{\varrho} (x_1 M_1 + x_2 M_2),$$

M_1 and M_2 being the molecular weight of the "1" and "2" component, respectively, and ϱ — the density of the mixture.

If, in particular, the "1" component is a non-dipolar gas whose molecules possess the quadrupole moment $\boldsymbol{H}_{(1)}$, the "2" component being a dipolar gas of molecules of dipolar moment $\mu_{(2)}$, the foregoing expressions yield (neglecting the tensor $B_{\alpha\beta:\gamma\delta}$)

$$(32)\quad A_Q^{(1)} = \frac{4\pi N}{15\, kT} \alpha_{(1)} \delta_{(1)} \boldsymbol{H}_{(1)}, \qquad A_Q^{(2)} = B_Q^{(22)} = 0,$$

$$(33)\qquad B_Q^{(11)} = \frac{2\pi^2 \alpha_{(1)} \delta_{(1)} H_{(1)}^3 N^2}{525 kT \sigma_{11}^5 \varepsilon_{11} y_{11}^2} \left\{ \alpha_{(1)} \delta_{(1)} H_8(y_{11}) + \frac{H_{(1)}^2 y_{11}^2}{6 \sigma_{11}^2 \varepsilon_{11}} H_{10}(y_{11}) \right\},$$

$$(34)\qquad B_Q^{(12)} = \frac{\pi^2 H_{(1)} \alpha_{(2)} \delta_{(2)} N^2}{75 kT \sigma_{12}^3 \varepsilon_{12} y_{12}^2} \left\{ \frac{H_{(1)}^2 \alpha_{(2)} \delta_{(2)}}{14 \sigma_{12}^2} H_8(y_{12}) + \right.$$

$$\left. + \frac{1}{30} \alpha_{(1)} \delta_{(1)} \mu_{(2)}^2 H_6(y_{12}) + \frac{H_{(1)}^2 \mu_{(2)}^2 y_{12}^2}{168 \sigma_{12}^2 \varepsilon_{12}} H_8(y_{12}) \right\},$$

with the usual combination rules [6]

$$\sigma_{12} = \frac{1}{2}(\sigma_1 + \sigma_2), \qquad \varepsilon_{12} = (\varepsilon_1 \varepsilon_2)^{1/2}.$$

The case of a carbon dioxide — methyl fluoride mixture will now be considered as an example. With the following data (see [1], [4], [6], [7]) for CO_2:

$$\alpha_{(1)} = 2.92 \times 10^{-24}\ \text{cm}^3, \qquad \delta_{(1)} = 0.27, \qquad H_{(1)} = -5 \times 10^{-26}\ \text{e.s.u.},$$

$$\sigma_{11} = 4 \times 10^{-8}\ \text{cm}, \qquad \varepsilon_{11/k} = 190^\circ\,\text{K},$$

and for CH_3F:

$$\alpha_{(2)} = 2.6 \times 10^{-24}\ \text{cm}^3, \qquad \delta_{(2)} = 0.11, \qquad \mu_{(2)} = 1.82 \times 10^{-18}\ \text{e.s.u.},$$

$$\sigma_{22} = 3.33 \times 10^{-8}\ \text{cm}, \qquad \varepsilon_{22/k} = 205^\circ\,\text{K},$$

Eqs. (33) and (34) yield, for $T = 300^\circ\text{K}$,

$$B_Q^{(11)} = -(1.04 + 25.61) \times 10^{-12} = -26.65 \times 10^{-12}\ \text{e.s.u. mole}^{-2},$$

$$B_Q^{(12)} = -(0.05 + 0.20 + 4.68) \times 10^{-13} = -0.49 \times 10^{-12}\ \text{e.s.u. mole}^{-2}.$$

By the foregoing evaluation, the contributions of the dipole-induced dipole and quadrupole-induced dipole interaction to the second virial coefficient B_Q are quite negligible when compared to those deriving from quadrupole-quadrupole or dipole-quadrupole interaction.

Finally, it should be noted that, similarly to Buckingham's equation (1) or to A_Q, investigation of the second virial coefficient B_Q can yield i.a. decisive information on the sign of the molecular quadrupole moment.

The author wishes to express his sincere thanks to Professor A. Piekara for his interest in the present paper and for his helpful discussions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

EXPERIMENTAL PHYSICS

On the Process of Domain Reorientation in Barium Titanate Single Crystals

by

J. STANKOWSKI

Presented by W. RUBINOWICZ on July 6, 960

Changes occurring in the domain structure of a $BaTiO_3$ single crystal on applying an electric field and one-dimensional pressure were investigated with a polarization microscope. The electric field and the pressure were applied in the direction of observation. Transparent gold or nickel electrodes obtained by evaporation *in vacuo*, were used.

The crystals used in the measurements presented a complex structure of dark and light domains. According to Merz [1], the dark domains are known as *c*-domains, the direction of polarization therein being perpendicular to the surface of the crystal. The light ones are *a*-domains, with polarization parallel to the surface. The blackening of the crystal is related to the direction of polarization in the domains.

a) One-dimensional pressure

When the crystal is subjected to one-dimensional pressure applied as shown in Fig. 1, the direction parallel to the crystal surface is privileged for the polarization *P*, resulting in lightening up of the dark domains. The polarization, hitherto directed perpendicularly to the surface of the crystal, now assumes a parallel

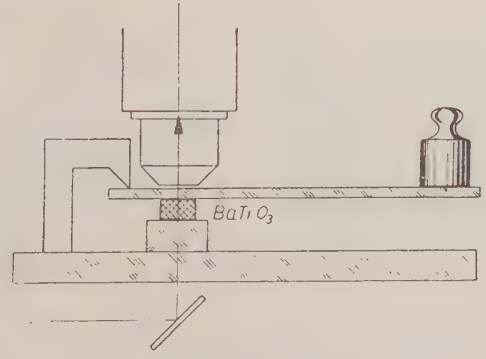

Fig. 1. Device for applying one-dimensional pressure parallel to the direction of observation

direction; the *c*-domains transform into *a*-domains. Figs. 2*a* and *b*, *c*, *d* illustrate this change in orientation of the polarization *P*. Fig. 2a is that of the crystal domain structure prior to the application of one-dimensional pressure; Fig. 2b is that of a sample under 10 atm. pressure. On removing the pressure, the dark domains increase. The process of increase of the *c*-domains is illustrated by Figs. 2c and 2d.

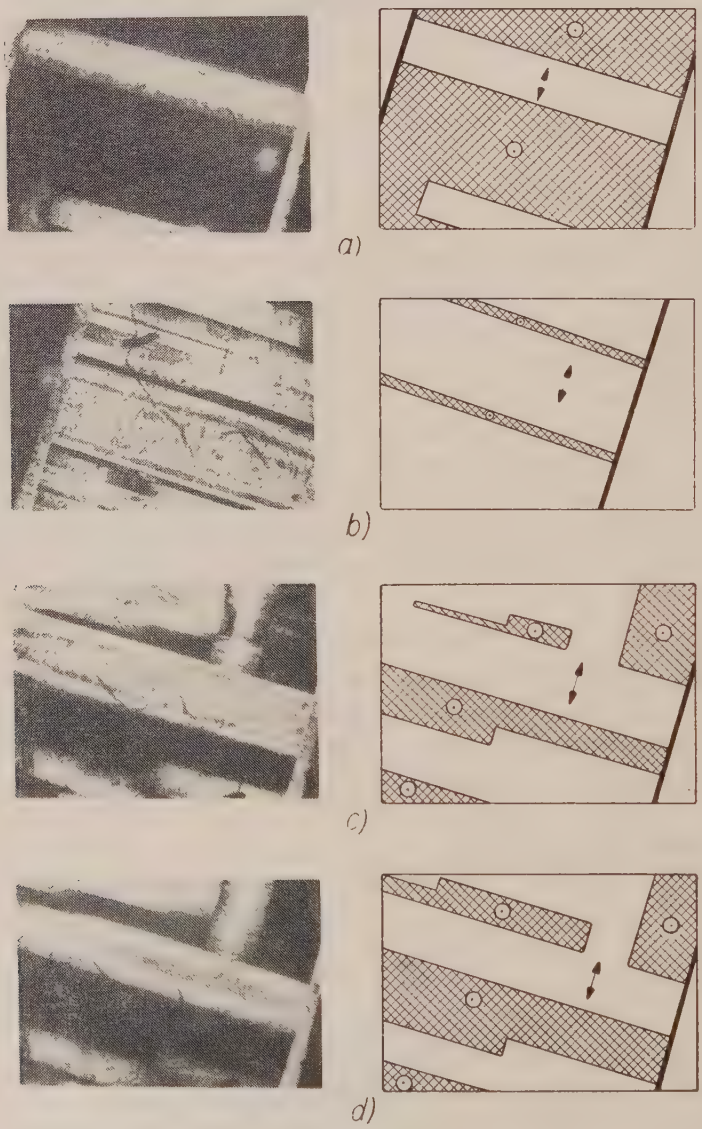

Fig. 2. Photographs of domain structure in a $BaTiO_3$ single crystal: *a*) prior to applying pressure *b*) on applying a 10 atm. pressure, *c*) one hour after pressure removal, *d*) 24 hours after pressur removal

b) Electric field

The method of applying an electric field to $BaTiO_3$ single crystals is shown schematically in Fig. 3. The crystal with electrodes obtained by evaporation was placed between two glass plates one surface of which was covered with a transparent electrode. The light emerged after passing through four evaporated layers and, consequently, the pictures of the domain structure are not so distinct as those discussed in section (a). On applying a d.c. field, the crystal becomes polarized,

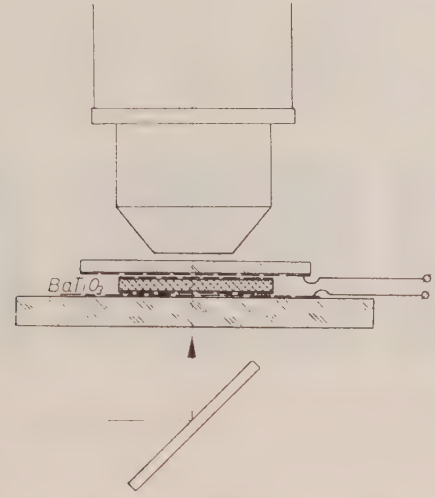

Fig. 3. Device for applying an electric field parallel to the direction of observation. Dashed lines indicate evaporated metal layers

i.e. the directions of polarization in the crystal are now parallel everywhere to the applied electric field. The polarized crystal examined under a microscope is black. In the process of switching the polarization part of the crystal lightens up; the reason for this is that switching proceeds not only by reversion of the polarization vector, but also by changing its direction. Thus, in addition to 180° switching, consecutive switchings at an angle of 90° occur. Fig. 4 schematically illustrates the process of switching in the central part thereof, the crystal is seen to light up.

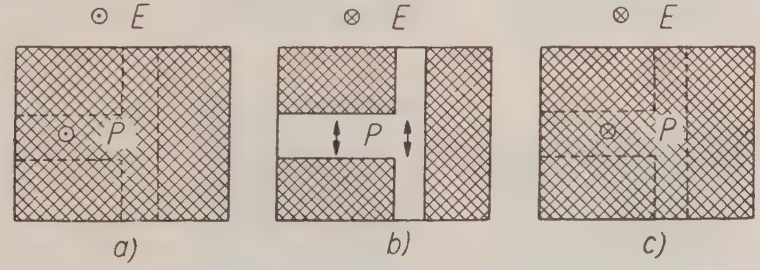

Fig. 4. Schematical illustration of lighting up of part of crystal in the switching process: *a*) $P \uparrow\uparrow E$, *b*) $P \leftarrow\downarrow E$, *c*) $P \downarrow\downarrow E$ following protracted effect of field

When the crystal is switched once, this effect of lighting up is of short duration and decays rapidly during a period of several seconds. It becomes accessible to observation as a permanent effect, however, if an a. c. and d. c. electric fields are applied simultaneously to the crystal. Fig. 5a illustrates the domain structure observed

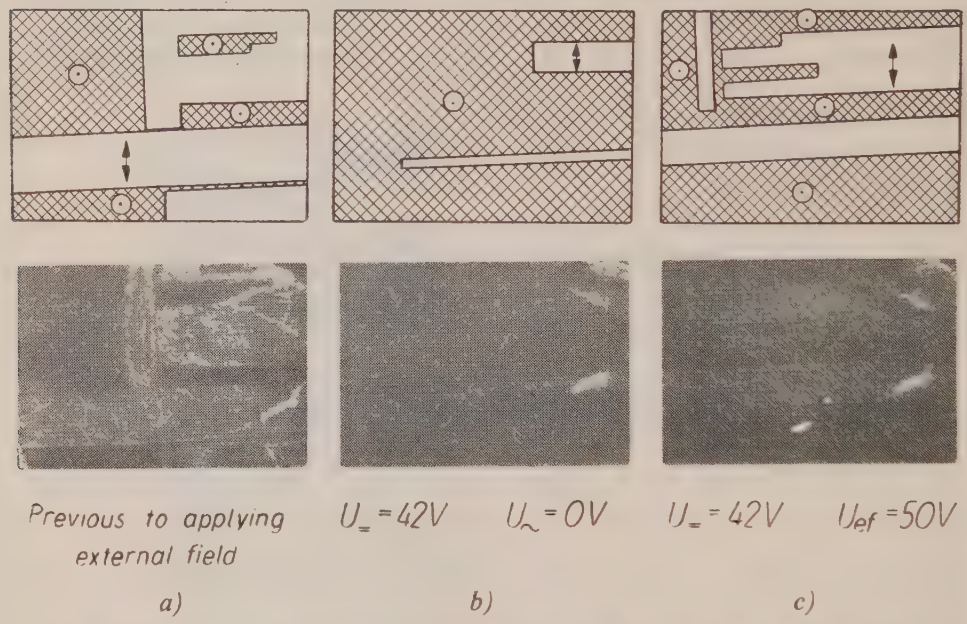

Fig. 5. Changes in domain structure produced by the effect of d. c. and a. c. fields

prior to applying a d. c. electric field. As soon as the latter is applied, the crystal becomes polarized and almost the whole of its surface is dark (Fig. 5b). An a. c. electric field is now applied, without removing the d. c. field. As long as the a. c. field strength is lesser than that of the d. c. field, $E_{\sim} < E_{=}$, the crystal remains dark.

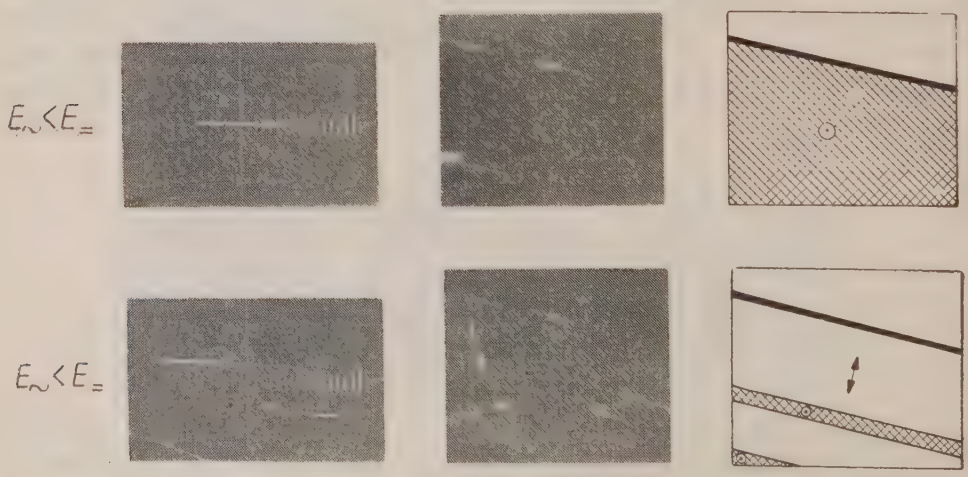

Fig. 6. Dielectric hysteresis loops and domain structure, for $E_{\sim} < E_{=}$ and $E_{\sim} > E_{=}$

When the a.c. field strength exceeds that of the d.c. field, $E_{\sim} > E_{=}$, part of the crystals lights up, as shown in Fig. 5c. The dielectric hysteresis loop of the crystal was taken simultaneously with observation of domain structures. The picture of this structure was found to light up simultaneously with the appearance of a dielectric hysteresis loop. Since the latter is directly related to domain reorientation, the effect of lightening up may be considered as resulting from domain reorientation according to the mechanism discussed. This is illustrated in Fig. 6.

Experimental results corroborate the switching mechanism proposed for $BaTiO_3$ by A. Piekara [2].

The author wishes to thank Professor A. Piekara for his valuable hints and discussions throughout the present investigation. The author is greatly indebted to Mrs H. Megaw for providing him with a $BaTiO_3$ single crystal.

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BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

Über die Konzentrationsabhängigkeit der Lumineszenzspektren von Organophosphoren

von

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Vorgelegt von A. JABŁOŃSKI am 14. Juli 1960

Die Untersuchungen der Kinetik der Phosphoreszenz von Organophosphoren wiesen bereits in den Zwanzigerjahren darauf hin, dass die Abklingungskurven der Phosphoreszenz Abweichungen vom einfachen Exponentialgesetz aufweisen können [13]. Die Untersuchungsergebnisse von mehreren Autoren [2], [5], [9], [12], erwiesen in den letzten Jahren, dass der Lumineszenzcharakter im hohen Grade von der Art der lumineszierenden Moleküle, sowie von der Art des Mediums in dem sich diese befinden, abhängt. Tiepliakow und Piatnitzkiï [11] führten Messungen der Konzentrationsabhängigkeit der Abklingungskurven durch, und stellten dabei fest, dass die Photolumineszenz gewisser Substanzen in einem weiten Konzentrationsbereich gemäss dem Exponentialgesetz abklingt, andere dagegen (in dem gleichen Konzentrationsbereich) von der Konzentration unabhängige Abweichungen vom Exponentialgesetz aufweisen. In den Jahren 1954 bis 1957 wurden theoretische Arbeiten von Jabłoński [6], [7] veröffentlicht, welche auf die Bestehungsmöglichkeit von solchen Organophosphoren hinweisen, für die die Abweichungen der Abklingungskurve vom Exponentialgesetz von der Konzentration abhängen. Wie es Frąckowiak [5] gezeigt hatte, gehören in Gelatine eingebetteten Akridingelb sowie Akridinorange zu dieser letzten Gruppe der Organophosphore. Er stellte fest, dass mit der Konzentrationssteigerung des Farbstoffes zusätzliche Gruppen von Lumineszenzzentren erscheinen, von denen jede durch eine bestimmte Abklingungskonstante charakterisiert wird; die Abklingungskurve eines hochkonzentrierten Organophosphors bildet infolgedessen eine Superposition von Exponentialkurven.

Der Zweck der vorliegenden Arbeit ist die Feststellung, ob irgendeine Abhängigkeit des Emissionsspektrums von der Konzentration der Farbstoffmoleküle für die Organophosphore bestehe, die eine konzentrationsabhängige Abweichung der Abklingungskurven von den Exponentialkurven aufweisen.

Es wurden Untersuchungen für Fluorescein in Borsäure durchgeführt. Abb. 2 stellt die Abklingungskurven für einige Konzentrationen im Bereich von $1{,}1 \cdot 10^{-6}$ g/g bis zu $1{,}1 \cdot 10^{-3}$ g/g dar. Aus diesen ist zu ersehen, dass der untersuchte Organo-

phosphor zu derjenigen Gruppe gehört, bei welcher die Abweichung von der Exponentialkurve, von der Konzentration abhängt. Die Abklingungskurven der Phosphoreszenz wurden unter Zuhlifenahme einer Versuchsanordnung bestimmt, bestehend aus einem Sekundärelektronen-Vervielfacher FEU-18, einem Kathodenstrahlenoszillograph, einem Kameraverschluss, einem gekreuzten Absorptionsfilterpaar,

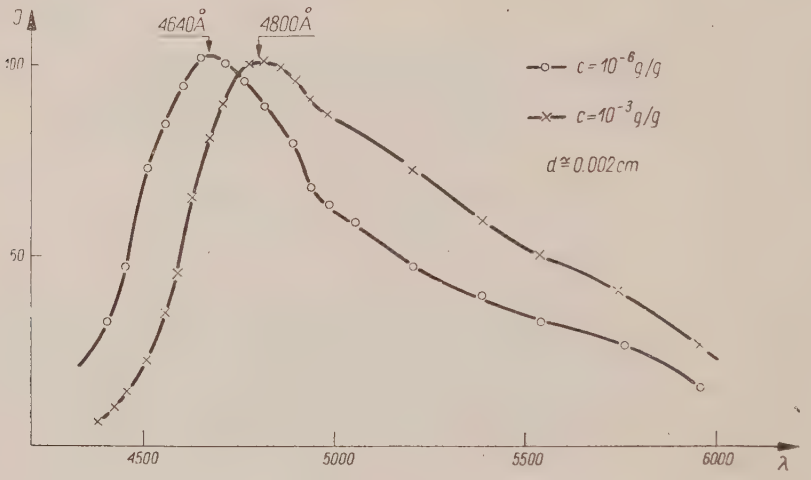

Abb. 1

sowie einem photographischen Apparat zur Feststellung des Abklingungsverlaufs am Film. Das Verfahren zur Ermittlung von Abklingungskurven ist von den Autoren in der Arbeit [1] angegeben worden.

Die Emissionsspektren der Photolumineszenz wurden nach der photoelektrischen Methode mit Hilfe des Sekundärelektronen-Vervielfachers FEU-18, sowie des Spiegelmonochromators von Zeiss, gemessen. Die Spalten des Monochromators wurden derart gewählt, dass im Bereich des gemessenen Spektrums die Spektralbreite des durchgelassenen Zichtes durchschnittlich 10 Å betrug. Infolge der Absorption wird der kurzwellige Teil des Emissionsspektrums verzerrt in einem vom Absorptionskoeffizienten des anregenden und des emittierten Lichtes abhängigen Grad. Die durch die Absorption nicht verzerrte Verteilung der emittierten Lichtintensitäten kann unter Anwendung der von Jabłoński [8] angegebenen Korrektur berechnet werden. Die Intensität der zum Monochromatorspalt gelungenen Photolumineszenz von der Wellenlänge λ ist:

$$(1) \qquad I_{\text{obs}}(\lambda, \lambda') = k\mu' \xi(\lambda, \lambda') I_0(\lambda') \int_0^d e^{-x\left(\frac{\mu'}{\cos\alpha} + \frac{\mu}{\cos\beta}\right)} dx$$

Es bezeichnen hier: μ' und μ entsprechend die Absorptionskoeffizienten des anregenden und des emittierten Lichtes, $\xi(\lambda, \lambda')$ die Photolumineszenzausbeute für λ mit der Wellenlänge λ' erregt, α den Brechungswinkel des einfallenden Strahlenbündels, β den Brechungswinkel des austretenden Strahlenbündels, K den von λ unabhängigen Proportionalitätsfaktor, $I_0(\lambda')$ die Intensität des primären Strahlen-

bündels, sowie d die Dicke des untersuchten Organophosphors. Nach der Integration erhalten wir:

$$(2) \qquad I_{obs}(\lambda, \lambda') = k\xi(\lambda, \lambda') I_0(\lambda') \frac{\cos\alpha}{1+\frac{\mu\cos\alpha}{\mu'\cos\beta}} \left[1 - e^{-d\left(\frac{\mu'}{\cos\alpha} + \frac{\mu}{\cos\beta}\right)}\right]$$

Da jedoch die Ausbeute proportional der Intensität des emittierten Lichtes ist, können wir aufschreiben:

$$(3) \qquad \xi(\lambda, \lambda') = C I_e(\lambda, \lambda')$$

wo C — der von λ unabhängiger Proportionalfaktor und $I_e(\lambda, \lambda')$ — die Intensität der durch die Absorption nicht verzerrten Photolumineszenz bezeichnen. Aus Gl. (2) und (3) erhalten wir:

$$(4) \qquad I_e(\lambda, \lambda') = \frac{I_{obs}(\lambda, \lambda')}{K \cdot C \cdot I_e(\lambda') \cos\alpha} \left(1+\frac{\mu\cos\alpha}{\mu'\cos\beta}\right) \left[1 - e^{-d\left(\frac{\mu'}{\cos\alpha} + \frac{\mu}{\cos\beta}\right)}\right]^{-1}$$

wo I_{obs} die beobachtete Intensitätsverteilung bedeutet. Die entsprechende Wahl der Konstante damit $K \cdot C \cdot I_0(\lambda') \cos\alpha = 1$ sei, führt zum Ausdruck:

$$(5) \qquad I_e = I_{obs}\left(1+\frac{\mu\cos\alpha}{\mu'\cos\beta}\right) \left[1 - e^{-d\left(\frac{\mu'}{\cos\alpha} + \frac{\mu}{\cos\beta}\right)}\right]^{-1}$$

Sind die Absorptionskoeffizienten μ und μ', die Dicke d des geprüften Phosphors sowie die entsprechenden Brechungswinkeln α, β bekannt, kann der Korrektionsfaktor berechnet werden, mit dem die entsprechende Intensität $I_{obs}(\lambda, \lambda')$ zu multi-

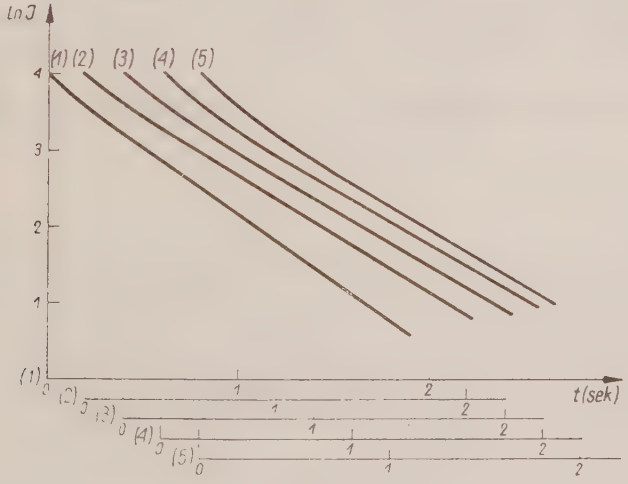

Abb. 2

plizieren ist, um ein unverzerrtes Emissionsspektrum zu erhalten. Abb. 1 stellt die, gemäss der Gleichung (5) korrigierte, Photolumineszenzverteilung für zwei verschiedene Konzentrationen, dar.

Nach Frąckowiak [5] führt die Konzentrationssteigerung zur Bildung von Lumineszenzzentren mit verschiedenen Anhäufungen von Farbstoffmolekülen,

was sich ausser der Änderung der Photolumineszenzpolarisation, auch durch den nichtexponentialen Verlauf der Abklingungskurven kennzeichnet. Seine Ergebnisse zusammen mit den in Abb. 2 dargestellten können zur Vermutung führen, dass die Konzentrationssteigerung von Fluorescein in der Borsäure die Generierung von Lumineszenzzentren verschiedener Art mit verschiedenen Abklingungskonstanten verursacht.

Aus den in Abb. 1 dargestellten Emissionsspektren ist eine deutliche Verschiebung des Emissionsmaximums der langwelligen Seite zu für starke Konzentrationen zu ersehen. Dieses Ergebnis, zusammengestellt mit der Tatsache, dass wir bei stärkeren Konzentrationen mit mehreren Zentrengruppen zu tun haben, führt zur Folgerung, dass jedes Lumineszenzzentrum ein Photolumineszenzlicht mit einer bestimmten für dieses charakteristischen spektralen Zusammensetzung, emittiert. Obige Folgerungen stimmen mit den Versuchsergebnissen von Chałupowskij [3] überein, der, indem er die Emissionsspektren der Phosphoreszenz von Fluorescein in Borsäure ($C = 5.10^{-4}$ g/g) in verschiedenen Zeitintervallen vom Moment der Unterbrechung der Anregung, untersuchte, in den verschiedenen Spektralbereichen verschiedenartigen Abklingungsverlauf feststellte. Die Abhängigkeit der Emissionsspektren der Photolumineszenz von der Konzentration der Farbstoffmoleküle bei unveränderten Absorptionsspektren*) bildet keinen Sonderfall, da bereits im Jahre 1955 Förster und Kasper [4] die Verschiebung der Fluoreszenzspektren von Pyren in Cyclohexan beobachtet hatten, was sie mit der Dimerisation der angeregten mit der nichtangeregten Moleküle, erklärten [11].

Die Autoren wünschen bei dieser Gelegenheit Herrn Professor A. Jabłoński ihren besten Dank für die mit ihnen geführten Diskussionen und das ihrer Arbeit erwiesene Interesse auszudrücken.

LABORATORIUM FUR PHOTOLUMINESZENZ, TORUŃ, POLNISCHE AKADEMIE DER WISSENSCHAFTEN.
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*) Im Fall von Fluorescein in Borsäure haben die Autoren keine sichtbaren Gestaltänderungen des Absorptionshauptstreifens bei der Konzentrationssteigerung des Farbstoffs bemerken können.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VIII, No. 9, 1960

EXPERIMANTAL PHYSICS

Depolarization of Fluorescence of Liquid Solutions

by

A. JABŁOŃSKI

Presented on July 25, 1960

In his well known theory of depolarization of fluorescence of solutions by rotational motion of dissolved luminescent molecules F. Perrin [8] assumes this motion to be governed by the Brownian laws of rotation of spherical particles suspended in the solvent. He obtains the following expression for the degree of polarization p of fluorescence excited by plane polarized primary light*) as a function of viscosity η and the absolute temperature T of the solution:

$$\frac{1}{p} - \frac{1}{3} = \left(\frac{1}{p_0} - \frac{1}{3}\right)\left(1 + \frac{kT}{v\eta}\tau\right), \tag{1}$$

p_0 — denoting the fundamental polarization (supposed to be a molecular constant), k — the Boltzmann constant, v — the volume of the luminescent molecule together with the solvation envelope, and τ — its mean life in the excited state.

In terms of emission anisotropy $r = \dfrac{2p}{3-p}$ the above expression reads [6]:

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \frac{kT}{v\eta}\tau\right). \tag{2}$$

It follows from Eqs. (1) and (2) that, for $T = \text{const.}$, $1/p$ and $1/r$ should be linear functions of $1/v\eta$, provided τ does not vary with η. In order to check this conclusion, experiments were performed in which mixtures of the two components were used as solvents, whose viscosities were varied by varying the relative concentrations of the components. In several instances marked deviations from linearity were observed, and especially in solutions of small viscosity, in which the degree of polarization becomes very low [10]. Considerations based on the theory of the Brownian rotation of ellipsoidal particles [9] have led Perrin to the conclusion that this discrepancy cannot be explained by asphericity of luminescent molecules, and that possibly the laws of the Brownian rotation cease to apply to molecules when the viscosity becomes very small [10].

*) Only the case of excitation by plane-polarized primary light in an absorption band connected with an electronic transition which is also responsible for emission of fluorescence is treated in this paper.

Further discrepances were discovered by Pringsheim and Vogels [12]. The fundamental polarization p_0 of the same luminescent molecule turned out to be different in different solvents approaching only in coloured gels to the theoretical value $p_0 = 0.5$, expected for linear oscillators. Moreover, only in cases in which the luminescent molecule is not soluble in one of the components of the solvent, $1/p$ appears to be a linear function of $1/v\eta$ [11], [12].

It may be also remarked that the deviations of the dependence of the mean duration of the components of fluorescence on polarization from theoretical curves observed by Kessel [7] point indirectly to the deficiency of the theory in question, the theoretical curves being calculated on its basis [4].

Thus it is clear that the theory of Perrin needs certain improvements, some of them being suggested below.

All the processes causing depolarization of photoluminescence may be divided into two groups: those whose effect depends on the decay time of photoluminescence, and those whose effect does not depend on it. The Brownian rotation of luminescent molecules belongs to the first group. Torsional vibrations of luminescent molecules may be quoted as an example of processes belonging to the second group. The depolarizing processes of the first group can be described by the equation

$$dr = -\varphi r\, dt, \tag{3}$$

where φ may be called depolarization probability per unit of time. If φ does not depend on time, there results

$$r(t) = r_0 e^{-\varphi t}, \tag{4}$$

where r_0 denotes the value of $r(t)$ at $t = 0$, i.e. at the moment of excitation. If there would be no depolarizing factors of the second group, r_0 would be the fundamental emission anisotropy of the luminescent molecule (usually it is not!).

For cases in which φ is a function of t we obtain

$$r(t) = r_0 e^{-\int_0^t \varphi dt} \tag{5}$$

instead of (4).

It follows from (4) that, provided the decay of total fluorescence obeys the exponential law (and $\varphi = \text{const.}$), the observed r by continuous excitation (i.e. the mean value of $r(t)$ over the whole decay time) is given by [6]:

$$r = \frac{r_0}{1+\varphi\tau}, \tag{6}$$

where τ is, as above, the mean life of the molecule in the excited state. Eq. (6) can be written as

$$\frac{1}{r} = \frac{1}{r_0}(1+\varphi\tau), \tag{7}$$

which is for $\varphi = kT/v\eta$ identical with Eq. (2).

From the fact that Eqs. (1) and (2) fail to describe adequately the observed facts, it may be inferred that the assumptions on which their derivations are based are insufficient. Some of the modifications needed are suggested by the following picture.

A luminescent molecule in a liquid solvent does not, as was assumed by Perrin, execute the Brownian rotation only. It executes in general torsional vibrations about its transient equilibrium orientation, only jumping from time to time into a new equilibrium orientation, if accidentally its vibrational energy becomes sufficient and the configuration of neighbouring solvent molecules favorable. The mean duration τ' of a certain equilibrium orientation, also called reorientation relaxation time, is in general much longer than the period τ' of torsional oscillations. According to Frenkel ([1], p. 250)

$$\tau' = \tau_0' e^{-\frac{E}{kT}}, \tag{7}$$

where E is the activation energy for reorientation. Only when kT approaches E the motion reduces to rotation of molecule.

Inspection of Eq. (4) shows that $q = 1/\tau'$ in the case of depolarization by reorientation, or, what is the same, by Brownian rotation. For larger molecules

$$\tau' = \frac{v\eta}{kT} \tag{8}$$

is generally considered as a sufficiently good approximation, although η appearing there may be not identical with the macroscopic viscosity (cf. e.g. [11], p. 375)*). The validity of (8) is provisionally assumed in this paper, and the improvement of the theory is looked for in the decay time independent depolarizing factors.

Depolarization by torsional vibrations was already considered in two papers [5]. This effect exists even in rigid solutions, in which practically no reorientations occur, although it is rather small in this case, and could be so far detected only for a rather light molecule (anthracene in methyl metacrylate) [2]. In liquid solutions the amplitude of torsional vibrations must be expected to be much larger and to produce an effect comparable with that produced by reorientation.

For a description of depolarization produced by torsional vibrations an expression may be used given by Perrin for the case of an isotropic solution, in which the transition dipols (linear virtual electronic oscillators) rigidly connected with luminescent molecules carry out vibrations about their directions of equilibrium [8]. In terms of emission anisotropy this expression reads [6]:

$$r_0' = 0.4 - 1.2\,u + 0.9\,u^2, \tag{9}$$

where u is the dispersion (mean square) of the sinus of the angle made by the transition dipol with its direction of equilibrium. For $u \ll 1$

$$r_0' \approx 0.4 - 1.2\,u \tag{10}$$

*) This picture is still oversimplified, as the possibility of existence of different reorientation relaxation times τ' and different periods τ_0' of torsional vibrations for different molecular axes is not taken into account.

is a sufficient approximation. For a given molecule u is a function of temperature and of torsion moments. These may vary or not, when the macroscopic viscosity is varied. They do not vary, if there is a solvation envelope whose composition does not change when the macroscopic viscosity of the solution is varied. In this case the depolarization produced by torsional vibrations does not depend on viscosity of the solution. If the composition of the solvation envelope depends on the relative concentration of the solvent's components, the moments of torsion acting on the luminescent molecule also depend on it. Thus, u can be considered in this case to be a function of macroscopic viscosity. Since its explicit form is not known, the best that we can do is to develop u in a Maclaurin series for constant temperature

$$u = u_0 + \frac{du}{d\left(\frac{1}{\eta}\right)} \frac{1}{\eta} + \dots \tag{11}$$

the first term of the series denoting u for $1/\eta = 0$.

Neglecting all terms but two and putting [11] into Eq. [10] we have

$$r_0'' = 0.4 - 1.2\,u_0 - 1.2 \frac{du}{d\left(\frac{1}{\eta}\right)} \frac{1}{\eta} = r_0'' - \frac{a}{\eta}.$$

Using this expression instead of r_0 in Eq. (2) we obtain

$$\frac{1}{r} = \frac{1 + \frac{kT}{v\eta}\tau}{r_0'' - \frac{a}{\eta}}. \tag{12}$$

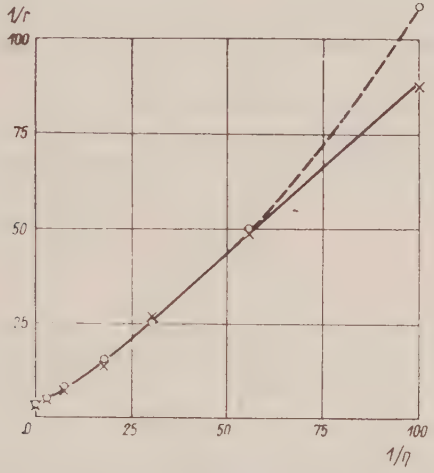
This expression for $1/r$ as a function of $1/\eta$ deviates from linearity in the same direction as the experimental curves of Perrin and those of Pringsheim and Vogels. The constants r_0'' and a may be chosen so as to obtain a best fit to experimental curves.

In the case of solvents composed of two components, in one of which the luminescent molecules are not soluble, the luminescent molecules are surrounded, as Pringsheim and Vogels [12] remark, by molecules of that component only, in which the luminescent molecules are soluble. Since this solvation envelope does not change when the relative concentration of the components is varied, $du/d(1/\eta) = a = 0$ in Eqs. (11) and (12) (the effect produced by torsional vibrations of the luminescent molecule together with its solvation envelope as a whole being negligible), and $1/r$ becomes a linear function of $1/\eta$, as observed by Pringsheim and Vogels. It follows also from Eq. (12) that the limiting value of r for $1/\eta = 0$ is in general smaller than the fundamental emission anisotropy r_0 ($= 0.4$ for linear oscillators considered in this paper) and may differ in different solvents, a fact also observed by Pringsheim and Vogels (12).

The limiting value may be further reduced, at least in some cases, by the following decay time independent process. A molecule may acquire, immediately after excitation (according to Franck-Condon principle), such an excess of vibrational energy that it may suffice to enhance the probability of reorientation. Such a process may be called "depolarization by initial shock". Its effect must depend on the wavelength of the exciting light.

The influence of quenching, reducing the mean life τ appearing in several equations above, is not discussed in this paper.

As an example, $1/r$ as a function of $1/\eta$ for uranin in glycerol + water is given in the Figure. The experimental points are taken from Pringsheim's book ([11], p. 372). The theoretical points are calculated by means of Eq. (12) with $\frac{kT}{v}\tau = 0.23$, $r_0'' = 0.359$ and $a = 0.00137$. The deviation of the last experimental point from the theoretical



$1/r$ as a function of $1/\eta$ for uranin in glycerol + water. × experimental points taken from Pringsheim [11], ○ points calculated by means of Eq. (12)

curve lies certainly within the limits of experimental errors, the accuracy of measurements of very small degrees of polarization ($p = 0.017$ in this case) being poor. The agreement of calculated points can be considered as satisfactory. However, a further refinement of the theory will probably be necessary for explaining some recently published observations [3], [13].

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1960

ТОМ VIII Резюме статей ВЫПУСК 9

В. ОРЛИЧ, **ОБ ИНТЕГРАЛЬНЫХ ПРЕДСТАВЛЕНИЯХ ЛИНЕЙНЫХ ФУНКЦИОНАЛОВ НА ПРОСТРАНСТВЕ φ-СУММИРУЕМЫХ ФУНКЦИЙ** стр. 563—565

В настоящей работе автор приводит представление посредством интеграла линейных функционалов на пространстве функций $L^{*\varphi}(a, b)$, определенном в работах [2[, [4] при дополнительном предположении, что функция φ удовлетворяет условию.

$$\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty.$$

Вид линейного функционала приводится в пкт. 1.5. настоящей работы.

В. ОРЛИЧ, **ОПЕРАЦИИ И ЛИНЕЙНЫЕ ФУНКЦИОНАЛЫ В ПРОСТРАНСТВАХ φ-ИНТЕГРИРУЕМЫХ ФУНКЦИЙ** . . . . . стр. 567—569

В работе автор предпринимает исследование φ-линейных функционалов над пространством $L^{*\varphi}(a, b)$ (дефиниция сравн. [3]), а в особенности рассматривает их интегральное представление.

Р. О. ГАНДИ, **ДОКАЗАТЕЛЬСТВО ПРЕДПОЛОЖЕНИЯ МОСТОВСКОГО** стр. 571—575

В работе приводится доказательство теоремы (доказанной также независимо Спектором, Fund. Math., **49** (1960) утверждающей, что каждый предикат вида $(a)\,A(a, a)$, где A — арифметическая формула, можно представить в виде $(E\beta)_{\text{Нур}}\,B(\beta, a)$, где B — арифметическая формула, а $(E\beta)_{\text{Нур}}$ обозначает: „существует гиперарифметическая функция".

Р. О. ГАНДИ, Г. КРЕЙЗЕЛЬ и В. В. ТАИТ, **О СУЩЕСТВОВАНИИ МНОЖЕСТВ** . . . . . стр. 577—582

Авторы доказывают, что если A — множество типа π_1^1 аксиом для арифметики, опирающейся на простой теории типов, то множества натуральных чисел, принадлежащих к каждой ω-модели аксиом A, идентичны с гиперарифметическими множествами.

Аналогичная теорема имеет также место в случае, когда A является рекуррентно перечислимым множеством аксиом. Однако, если выражение „гиперарифметическое" заместить выражением „рекуррентное", тогда следует говорить вместо — о множествах, принадлежащих к любой ω-модели — о множествах натуральных чисел, которые для каждой модели репрезентированы некоторым элементом этой модели.

А. ЭРЕНФОЙХТ и М. ФИШ, **УСЛОВИЯ НЕОБХОДИМЫЕ И ДОСТАТОЧНЫЕ ДЛЯ СЛАБОГО ЗАКОНА БОЛЬШИХ ЧИСЕЛ** . . . . стр. 583—585

Теорема. *Пусть* $\{x_i\}$ $(i = 1, 2, \ldots)$ *будет последовательностью независимых и одинаково распределенных случайных величин. Соотношение*

$$\lim_{n\to\infty} P\left(\left|\frac{x_i + \cdots + x_n}{n} - a\right| > \varepsilon\right) = 0$$

имеет место при любом $\varepsilon > 0$, *тогда и только тогда, когда характеристическая функция* $\varphi(t)$ *случайной величины* x_i *дифференцируема в точке* $t = 0$ *и притом* $\varphi'(0) = ai$.

С. СВЕРЧКОВСКИЙ, **ОБ ИЗОМОРФНЫХ СВОБОДНЫХ АЛГЕБРАХ** стр. 587—588

В работе содержатся теоремы о свободных алгебрах с неодинаковым числом образующих. Кроме того, доказывается, что всякая арифметическая прогрессия является множеством мощностей всех базисов некоторой алгебры.

И. ЛОЙЧИК-КРУЛИКЕВИЧ, **ЗАМЕТКА О ГРАНИЧНЫХ СВОЙСТВАХ РЕШЕНИЙ** . . . . . стр. 597—603

Пусть G — ограниченная область пространства E^m с переменными $x_1, \ldots, x_m$. Обозначим через $\bar{G}$ замыкание области G, а через $\dot{G}$ — ее край. Топологическое произведение $Gx(0, \infty)$ обозначим через D. Боковую поверхность этой области мы обозначили через σ.

Рассматривается почти линейное уравнение (1), в котором X является точкой пространства E^m с координатами $x_1, \ldots, x_m$. Полагается, что правая сторона этого уравнения обладает следующими свойствами:

(B) $f(X,t,z)$ определена в $Dx(-\infty,\infty)$; упомянутая функция в этом множестве удовлетворяет следующему условию:

для каждой пары $z'<z''$ имеется неравенство:

(5) $$f(X,t,z')\leqslant f(X,t,z'').$$

Коэффициенты уравнения (1) подчинены следующему предположению:

(C) $a_{ij}(X,t)$, $b_k(X,t)$ — определены, а характеристическая форма выражения $\sum_{ij=1}^{m} a_{ij}\lambda_i\lambda_j$ — определена положительно в D.

При таких предположениях, опираясь на вспомогательную Теорему 1, автор дает доказательство справедливости следующей Теоремы:

Теоорема 2. *Предполагается, что выполнены условия* (B) *и* (C); *предполагаеотся также, что*

(D) *функция* $f(X,t,z)$, *будучи непрерывной по отношению к* $(X,t)\epsilon D$, *выполняет уравнение*

(10) $$\lim_{t\to\infty} f(X,t,0)=0,$$

равномерно по отношению к $X\epsilon\overline{G}$, *причем* $f(X,t,0)\not\equiv 0$) *в* D (*случай, когда* $f(X,t,0)\equiv 0$ *рассматривается в работе* [5]);

предполагается также, что

(E) *имеется показатель* $i\leqslant m$ *такой что* $a_{ii}(X,t)\geqslant a_0>0$ *и* $b_i(X,t)\leqslant b_0$, *где* $b_0>0$ *для* $(X,t)\epsilon D$.

Тогда решение уравнения (1) регулярное в D, *которое при* $t\to\infty$ *стремится равномерно к нулю по отношению к* $X\epsilon\dot{G}$, *стремится также равномерно к нулю и во всей области* $\overline{G}$.

Если положить, что коэффициенты уравнения (1) ограничены следующим образом:

(A) $$\lim_{t\to\infty} a_{ij}(X,t)=a_{ij}(X)\ \text{и}\ \lim_{t\to\infty} b_k(X,t)=b_k(X)$$

причем эта сходимость равномерна по отношению к $X\epsilon\overline{G}$, то мы имеем дело также с уравнением эллиптического типа (2).

В теоремах 3 и 4, а также и в замечании 1 указаны условия достаточные для того, чтобы решение $u(X,t)$ уравнения (1), регулярное в D при $t\to\infty$ стремилось равномерно к решению эллиптического уравнения (2).

Если предположения, касающиеся функции $f(X,t,u,u'_{x_1},\ldots,u'_{x_m})$ сделать более сильными, тогда возможно доказательство аналогичной теоремы также и для такого уравнения.

Рассматривается также случай третьей проблемы Фурье и доказывается справедливость аналогичных теорем для би-регулярного решения уравнения (1), удовлетворяющего краевому условию (18).

М. РОХОВСКИЙ, **О ПРЕДСТАВЛЕНИИ МНОГООБРАЗИЯ В ЕВКЛИДОВОМ ПРОСТРАНСТВЕ КАК ВЫПУКЛОЙ ГИПЕРПОВЕРХНОСТИ** . . стр. 605—608

Обозначим через M^n n-мерное замкнутое, ориентируемое многообразие, класса C^∞. Пусть $x: M^n \to E^{n+1}$ является регулярным отображением, т.е. таким, что матрица Якоби этого отображения имеет ранг n во всякой точке $p \in M^n$, класса C^∞, многообразия M^n в $n+1$-мерное евклидово пространство E^{n+1}. Скажем, что квадратичная форма неположительна если она всегда принимает отрицательные значения или нуль.

В работе доказывается следующая теорема, являющаяся обобщением теоремы Адамара ([1], стр. 352):

Если вторая квадратичная форма гиперповерхности $x(M^n)$ всюду неположительна, то многообразие M^n погружено в E^{n+1} как выпуклая гиперповерхность.

К. БОРСУК, **НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ПОЛОЖЕНИИ МНОЖЕСТВ В ПРОСТРАНСТВЕ** . . . . . стр. 609—613

Замкнутое подмножество A пространства X называется его гомотопическим k-остовом, если его размерность $\leqslant k$ и если каждое замкнутое подмножество B пространства X размерности $\leqslant k$ переводимо — путем гомотопии в X — на некоторое подмножество множества A. В равоте дается доказательство, что для каждого пространства ANR, удовлетворяющего так наз. условию (Δ) имеется гомотопический k-остов.

К. БОРСУК, **О НЕКОТОРОМ ОБОБЩЕНИИ ГРУПП КОГОМОТОПИИ** стр. 615—620

Известно, что классы гомотопии преобразований плотного пространства A размерности $k \leqslant 2n-1$ в n-мерную сферу S образуют группу $\pi^n(A)$ (так наз. n-тую группу когомотопии пространства A), в которой групповой операцией является так наз. гомотопическое складывание преобразований. Если A является так наз. гомотопическим k-остовом пространства X (в смысле предшествующей работы), то оказывается, что классы гомотопии частичных преобразований f/A, (где $f \in (S^n)$ при операции гомотопического складывания порождают группу $\pi_k^n(X)$ причем ее альгебраическая структура независима от метода выбора k-остава пространства A; эта структура зависит лишь от X, k и n.

В особом случае, когда $k = \dim X < R_n - 1$, группа $\pi_k^n(X)$ совпадает с группой $\pi^n(X)$.

К. МОРЭН и Л. МОРЭН, **ЯДЕРНОСТЬ НЕКОТОРЫХ ВЛОЖЕНИЙ РЕЛЛИХА - СОБОЛЕВА** . . . . . стр. 621—624

В работе авторы доказывают ядерность вложений пространств Соболева $H_0^{m+k}(\Omega_n) \to H_0^k(\Omega_n)$ для $m > n$, $k \geqslant 0$. В качестве применения дается уточнение теоремы Гординга-Березанского об обобщенных собственных функциях дифференциальных операторов.

Я. КРЖИЖ, **ДИСТОРСИОННЫЕ ТЕОРЕМЫ ДЛЯ СВЯЗНЫХ ВЫПУКЛЫХ ФУНКЦИЙ** . . . . . стр. 625—627

Путь C_M обозначает класс функций вида $f(z) = a_1 z + a_2 z + \ldots, |a_1| = 1$, одновалентных в круге $K(0,1)$ с центром 0 и радиусом 1, отображающих $K(0,1)$ на выпуклую область $\Omega(f)$, содержащуюся в круге $K(0,M)$ где $M > 1$ фиксировано.

Доказывается, что радиус самого большого круга $K(0,\delta)$, содержащегося во всех областях $\Omega(f)$, где $f \in C_M$, выражается формулой $\delta = M \sin\Theta$, где Θ является единственным решением уравнения (1, 2), удовлетворяющим условию $0 < \Theta < \pi/2$.

Кроме того, приводятся границы нижнего и верхнего модуля $|f(z)|$ для всех $f \in C_M$.

В. ГАНУС, **О СООТНОШЕНИЯХ, ОПРЕДЕЛЯЮЩИХ ИНТЕНСИВНОСТЬ ЛИНИЙ В ДУБЛЕТНЫХ ДУГОВЫХ СПЕКТРАХ АЛЮМИНИЯ, ГАЛЛИЯ, ИНДИЯ И ТАЛЛИЯ** . . . . . стр. 629—636

Формулы для относительных интенсивностей линий в дублетных спектрах элементмв третьей группы (для главной, резкой и диффузной серии) получены в результате обобщения вычисления Ферми для главной серии щелочных металлов с учетом взаимодействия спин-орбита. Вычисления обосновываются на значениях радикальных интегралов, приводимых в таблицах Бэтса и Дамгаарда. Результаты для главной серии вообще находятся в хорошем согласии с опытными данными, однако, в случае резкой и диффузной серий, получаются значительные расхождения. Автором предлагается объяснение последних.

С. КЕЛИХ, **ТЕОРИЯ ДВОЙНОГО ЛУЧЕПРЕЛОМЛЕНИЯ, ВЫЗЫВАЕМОГО В ГАЗОВОЙ СМЕСИ ГРАДИЕНТОМ ЭЛЕКТРИЧЕСКОГО ПОЛЯ** стр. 637—644

Автор работы представляет теорию двойного лучепреломления, вызываемого градиентом электрического поля в мультикомпонентной газовой смеси. Молекулярная константа Q_m, характеризующая этот эффект, представлена в виде вириального разложения $Q_m = A + B/V_m + \ldots$. Второй вириальный коэффициент этого разложения вы-

числен для смеси полярных газов, причем, кроме потенциала Леннарда-Джонса, учитывается энергия мультипольного взаимодействия между молекулами.

Теория применяется к двухкомпонентному газу; в качестве примера рассматривается смесь CO_2-CH_3F.

Я. СТАНКОВСКИЙ, **ПРОЦЕСС ПЕРЕОРИЕНТИРОВКИ ДОМЕН В МОНОКРИСТАЛЛАХ ТИТАНАТА БАРИЯ** . . . . . стр. 645—649

При применении поляризационного микроскопа автор исследовал изменения структуры домен в монокристаллах $BaTiO_3$, вызываемые постоянным электрическим полем и одномерным давлением.

Найдено, что в процессе переполяризования, кроме изменений скачков на 180°, имеют место изменения поляризации на 90°. Одновременно с наблюдением доменной структуры были произведены снимки петли гистерезиса. Оказалось, что прояснение частей кристалла, являющееся следствием переориентировки домен на 90°, имеет место одновременно с появлением петли гистерезиса.

А. БОНЧИНСКИЙ и М. ЧАЙКОВСКИЙ, **ЗАВИСИМОСТЬ СПЕКТРОВ ЛЮМИНЕСЦЕНЦИИ ОТ КОНЦЕНТРАЦИИ МОЛЕКУЛ КРАСИТЕЛЯ В ОРГАНИЧЕСКИХ ФОСФОРАХ** . . . . . стр. 651—654

Произведены измерения спектров эмиссии (излучения) флуоресцеина в борной кислоте в зависимости от концентрации молекул красителя.

Констатировано, что максимумы спектров эмиссии (излучения) перемещаются в направлении длинных волн по мере роста концентрации при неизменной форме спектров поглощения. Учитывается влияние реасорбции (вторичного поглощения) на спектр излучения, применяя исправление, введенное А. Яблонским (C. R. de Séances, de la Soc. Pol. de Phys., 7, 1926).

А. ЯБЛОНСКИЙ, **ДЕПОЛЯРИЗАЦИЯ ФЛУОРЕСЦЕНЦИИ ЖИДКИХ РАСТВОРОВ** . . . . . стр. 655—660

Теория Перрина вращательной деполяризации флуоресценции растворов пополняется автором благодаря введению новых факторов деполяризации как торсионные колебания и другие.

Пополненная таким образом теория Перрина делает возможным объяснить — по крайней мере качественно — некоторые отклонения от теории Перрина, которые наблюдали Перрин, а также Прингсгейм и Фогельс.

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